

Quantum Mechanics - 1: HW 2 Solutions

Leo Radzihovsky
Paul Martens

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1 Problem 1

Showing that the Schrödinger's equation contains Hamilton-Jacobi equation (in the $\hbar \rightarrow 0$ limit) and the continuity equation. We start with the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\partial_t\Psi \quad (1)$$

we suppose $\Psi = |\Psi| e^{\frac{iS}{\hbar}}$. We'll need the derivatives

$$\nabla\Psi = (\nabla|\Psi|)e^{\frac{iS}{\hbar}} + |\Psi|\frac{i}{\hbar}\nabla S e^{\frac{iS}{\hbar}} \quad (2)$$

$$\nabla^2\Psi = \nabla^2|\Psi|e^{\frac{iS}{\hbar}} + \frac{2i}{\hbar}\nabla|\Psi|\nabla S e^{\frac{iS}{\hbar}} + \frac{i}{\hbar}|\Psi|\nabla^2 S e^{\frac{iS}{\hbar}} - \frac{1}{\hbar^2}|\Psi|(\nabla S)^2 e^{\frac{iS}{\hbar}} \quad (3)$$

$$i\hbar\partial_t\Psi = i\hbar\partial_t|\Psi|e^{\frac{iS}{\hbar}} - |\Psi|\partial_t S e^{\frac{iS}{\hbar}} \quad (4)$$

Using the above we can write (1)

$$-\frac{\hbar^2}{2m}\nabla^2|\Psi| - \frac{i\hbar}{m}\nabla|\Psi|\cdot\nabla S - \frac{i\hbar}{2m}|\Psi|\nabla^2 S + \frac{1}{2m}|\Psi|(\nabla S)^2 + V|\Psi| = i\hbar\partial_t|\Psi| - |\Psi|\partial_t S \quad (5)$$

both real and imaginary parts of this equation must be satisfied, so we can look at these parts independently.

1.1 real:

$$-\frac{\hbar^2}{2m}\nabla^2|\Psi| + \frac{|\Psi|}{2m}(\nabla S)^2 + V|\Psi| = -|\Psi|\partial_t S \quad (6)$$

In the classical limit $\hbar \rightarrow 0$ this becomes the H-J equation.

$$\frac{1}{2m}(\nabla S)^2 + V = -\partial_t S \quad (7)$$

1.2 Im:

$$-\frac{\hbar}{m} \nabla |\Psi| \cdot \nabla S - \frac{\hbar}{2m} |\Psi| \nabla^2 S = \hbar \partial_t |\Psi| \quad (8)$$

to get the continuity equation we multiply both sides by $2|\Psi|$, and note that $\partial |\Psi|^2 = 2|\Psi| \partial |\Psi|$.

$$\partial_t |\Psi|^2 + \nabla \cdot \left(|\Psi|^2 \frac{\nabla S}{m} \right) = 0 \quad (9)$$

from the above equation we can identify $P = |\Psi|^2$ as the probability density,

and $\vec{J} = \underbrace{|\Psi|^2}_{\text{density}} \underbrace{\frac{\vec{\nabla} S}{m}}_{\text{velocity}}$ as the probability current.

2 Problem 2

Show that $\hat{U} = e^{-\frac{i}{\hbar} \hat{H} t}$ satisfies the Schrödinger equation.

2.1 generally

$$i\hbar \partial_t \hat{U} = i\hbar \left(\frac{-i}{\hbar} \right) \hat{H} e^{-\frac{i}{\hbar} \hat{H} t} = \hat{H} \hat{U} \quad (10)$$

so \hat{U} satisfies the Schrödinger equation.

$$i\hbar \partial_t \hat{U} = \hat{H} \hat{U} \quad (11)$$

2.2 An explicit case

For a free particle

$$U(x, x'; t) = \sqrt{\frac{m}{2\pi i \hbar t}} e^{i \frac{m(x-x')^2}{2\hbar t}} \quad (12)$$

therefore

$$i\hbar \partial_t U = \sqrt{\frac{m}{2\pi i \hbar}} i\hbar \left[-\frac{t^{-\frac{3}{2}}}{2} - it^{-\frac{1}{2}} \frac{m(x-x')^2}{2\hbar t^2} \right] e^{i \frac{m(x-x')^2}{2\hbar t}}$$

and

$$\begin{aligned} -\frac{\hbar^2}{2m} \partial_x^2 U &= -\frac{\hbar^2}{2m} \sqrt{\frac{m}{2\pi i \hbar}} t^{-\frac{1}{2}} \partial_x \left(\frac{im}{\hbar t} (x-x') e^{i \frac{m(x-x')^2}{2\hbar t}} \right) \\ &= -\frac{\hbar^2}{2m} \sqrt{\frac{m}{2\pi i \hbar}} t^{-\frac{1}{2}} \left(\frac{im}{\hbar t} - \left(\frac{m}{\hbar t} \right)^2 (x-x')^2 \right) e^{i \frac{m(x-x')^2}{2\hbar t}} \\ &= \sqrt{\frac{m}{2\pi i \hbar}} i\hbar \left[-\frac{t^{-\frac{3}{2}}}{2} - it^{-\frac{1}{2}} \frac{m(x-x')^2}{2\hbar t^2} \right] e^{i \frac{m(x-x')^2}{2\hbar t}} \end{aligned}$$

Therefore our example for U is a solution to the Schrödinger equation

3 Problem 3

The spin = 1 representation of the $SU(2)$ group can be written

$$L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; L_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (13)$$

3.1 possible values of L_z

The possible values of L_z are just it's eigen values. In this representation L_z is diagonal so the eigen values can be just read off the diagonal.

$$\pm 1, 0$$

3.2 Finding the variance ΔL_x of $|L_z = 1\rangle$

3.2.1 The Paul Way

I like to just multiply out the matrices. This should be real easy because $|L_z = 1\rangle$ is just $(1, 0, 0)$, so $\langle A \rangle$ is just the upper left number in the matrix A . We'll need to know L_x^2 .

$$L_x^2 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad (14)$$

So,

$$\langle L_x \rangle = \frac{1}{\sqrt{2}} (1 \ 0 \ 0) \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \quad (15)$$

and

$$\langle L_x^2 \rangle = \frac{1}{2} (1 \ 0 \ 0) \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \quad (16)$$

The variance is defined

$$\langle L_x^2 \rangle - \langle L_x \rangle^2 = \frac{1}{2} \quad (17)$$

3.2.2 The Leo Way

Another method is find the eigenvectors of the matrix L_x . This will answer part c as well. First we find the eigen values.

$$\begin{vmatrix} -\lambda & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\lambda & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\lambda \end{vmatrix} = -\lambda^3 + \lambda = 0 \quad (18)$$

This has solutions $\lambda = -1, 0, 1$. Eigenvectors are pretty easy to find.

$$|L_x = 0\rangle = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (19)$$

we can find $|L_x = \pm 1\rangle$ by solving the system

$$\pm L_x^+ + \frac{\sqrt{2}}{2} L_x^0 = 0 \quad (20)$$

$$\frac{\sqrt{2}}{2} L_x^+ \pm L_x^0 + \frac{\sqrt{2}}{2} L_x^- = 0 \quad (21)$$

$$\frac{\sqrt{2}}{2} L_x^0 \pm L_x^- = 0 \quad (22)$$

we also want values for L_x^0, L_x^\pm so that $(L_x^0)^2 + (L_x^+)^2 + (L_x^-)^2 = 1$. The solutions to these equation are

$$|L_x = \pm 1\rangle = \frac{1}{2} \begin{pmatrix} \pm 1 \\ \sqrt{2} \\ \pm 1 \end{pmatrix} \quad (23)$$

Now that we know the eigenvectors in the L_x basis we can write $|L_z = 1\rangle$ in the L_x basis

$$|L_z = 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2} \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} \quad (24)$$

so,

$$|L_z = 1\rangle = \frac{1}{2} |L_x = 1\rangle + \frac{\sqrt{2}}{2} |L_x = 0\rangle - \frac{1}{2} |L_x = -1\rangle \quad (25)$$

Now calculating $\langle L_x \rangle$ is really easy.

$$\langle L_x \rangle = (1) |\langle L_x = 1 | L_z = 1 \rangle|^2 + (-1) |\langle L_x = -1 | L_z = 1 \rangle|^2 = \frac{1}{4} - \frac{1}{4} = 0. \quad (26)$$

And

$$\langle L_x^2 \rangle = (1)^2 |\langle L_x = 1 | L_z = 1 \rangle|^2 + (-1)^2 |\langle L_x = -1 | L_z = 1 \rangle|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad (27)$$

so

$$\Delta L_x^2 = \langle L_x^2 \rangle - \langle L_x \rangle^2 = \frac{1}{2} \quad (28)$$

3.3 finding the normalized L_x eigenvectors

We did this above they were

$$\begin{aligned} |L_x = 0\rangle &= \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ |L_x = \pm 1\rangle &= \frac{1}{2} \begin{pmatrix} \pm 1 \\ \sqrt{2} \\ \pm 1 \end{pmatrix} \end{aligned} \quad (29)$$

3.4 given that we are in the $L_z = 1$ what is $P(L_x = 0, \pm 1)$

We have $|L_x\rangle$ in the L_z basis so we can basically read the probabilities off.

$$P_{L_z=1 \rightarrow L_x=\pm 1} = |\langle L_x = \pm 1 | L_z = 1 \rangle|^2 = \frac{1}{4} \quad (30)$$

$$P_{L_z=1 \rightarrow L_x=0} = |\langle L_x = 0 | L_z = 1 \rangle|^2 = \frac{1}{2} \quad (31)$$

3.5 collapse of wavefunctions

we started with the wavefunction

$$|\Psi\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} \quad (32)$$

we find that $L_z^2 = 1$, so that means the only $L_z = \pm 1$ is possible. So our state after is.

$$|\Psi_{after}\rangle = \underbrace{\left(\frac{1}{2} |L_z = 1\rangle + \frac{\sqrt{2}}{2} |L_z = -1\rangle \right)}_{L_z \neq 0} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}} \quad (33)$$

$$|\Psi_{after}\rangle = \frac{1}{\sqrt{3}} |L_z = 1\rangle + \sqrt{\frac{2}{3}} |L_z = -1\rangle \quad (34)$$

The probability that we get $L_z^2 = 1$ as an outcome can be read off the state.

$$\underbrace{\frac{1}{4}}_{L_z=1} + \underbrace{\frac{1}{2}}_{L_z=-1} = \frac{3}{4} \quad (35)$$

The probabilities of an L_z measurement afterwards can be read off the final state.

$$P(L_z = 1) = \frac{1}{3} \quad (36)$$

$$P(L_z = 0) = 0 \quad (37)$$

$$P(L_z = -1) = \frac{2}{3} \quad (38)$$

3.6 Probabilities to states

Suppose we have a state

$$|\Psi\rangle = C_+ |L_z = 1\rangle + C_0 |L_z = 0\rangle + C_- |L_z = -1\rangle \quad (39)$$

we know that

$$|C_+|^2 = \frac{1}{4} \quad (40)$$

$$|C_0|^2 = \frac{1}{2} \quad (41)$$

$$|C_-|^2 = \frac{1}{4} \quad (42)$$

The most general solutions to these are

$$C_n = \sqrt{P_n} e^{i\delta_n} \quad (43)$$

These phases are physical and can be measured by considering $L_x = 0$.

$$\begin{aligned} P(L_x = 0) &= |\langle L_x = 0 | \Psi \rangle|^2 \\ &= \left| \frac{e^{i\delta_1}}{2} \langle L_x = 0 | L_z = 1 \rangle + \frac{e^{i\delta_2}}{\sqrt{2}} \langle L_x = 0 | L_z = 0 \rangle + \frac{e^{i\delta_3}}{2} \langle L_x = 0 | L_z = -1 \rangle \right|^2 \\ &= \left| \frac{1}{2} e^{i\delta_1} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} e^{i\delta_2} 0 + \frac{1}{2} e^{i\delta_3} \left(-\frac{\sqrt{2}}{2} \right) \right|^2 \\ &= \left| \frac{1}{2^{3/2}} (e^{i\delta_1} - e^{i\delta_3}) \right|^2 \\ &= \frac{1}{2} \sin^2 \left(\frac{\delta_1 - \delta_3}{2} \right) \end{aligned} \quad (44)$$

4 Problem 4

$$\vec{L} = \vec{r} \times \vec{p} = \epsilon_{ijk} r_i p_j \quad (45)$$

4.1 Poisson bracket

$$\begin{aligned} \{L_i, L_j\} &= \{\epsilon_{ikl} x_k p_l, \epsilon_{jmn} x_m p_n\} \\ &= \epsilon_{ilk} \epsilon_{jmn} \{x_l p_k, x_m p_n\} \\ &= \epsilon_{ilk} \epsilon_{jmn} \left[\underbrace{\frac{\partial(x_l p_k)}{\partial x_q}}_{\delta_{lq} \delta_{nq} p_k x_m} \frac{\partial(x_m p_n)}{\partial p_q} - \underbrace{\frac{\partial(x_l p_k)}{\partial p_q}}_{\delta_{kq} \delta_{mq} x_l p_n} \frac{\partial(x_m p_n)}{\partial x_q} \right] \\ &= \epsilon_{ilk} \epsilon_{jml} p_k x_m - \epsilon_{ilk} \epsilon_{jkn} x_l p_n \\ &= p_j x_i - p_m x_m \delta_{ij} - p_i x_j + \delta_{ij} x_n p_n \\ \{L_i, L_j\} &= x_i p_j - x_j p_i = \epsilon_{ijk} L_k \end{aligned} \quad (46)$$

4.2 commutation relations

$$\begin{aligned}
[\hat{L}_i, \hat{L}_j] &= [\hat{x}_l \hat{p}_k, \hat{x}_m \hat{p}_n] \epsilon_{ilk} \epsilon_{jmn} \\
&= \left(\underbrace{\hat{x}_l [\hat{p}_k, \hat{x}_m \hat{p}_n]}_{-i\hbar \delta_{km} \hat{p}_n} + \underbrace{[\hat{x}_l, \hat{x}_m \hat{p}_n] \hat{p}_k}_{i\hbar \delta_{ln} \hat{x}_m} \right) \epsilon_{ilk} \epsilon_{jmn} \\
&= i\hbar (\hat{x}_m \hat{p}_k \epsilon_{ink} \epsilon_{jmn} - \hat{x}_l \hat{p}_n \epsilon_{ilk} \epsilon_{jkn}) \\
&= i\hbar [\hat{x}_m \hat{p}_k (\delta_{kj} \delta_{im} - \delta_{kn} \delta_{ij}) - \hat{x}_l \hat{p}_n (\delta_{in} \delta_{lj} - \delta_{ij} \delta_{ln})] \\
&= i\hbar (\hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i) \\
[\hat{L}_i, \hat{L}_j] &= i\hbar \epsilon_{ijk} L_k
\end{aligned} \tag{47}$$

5 Problem 5

Finding the energy eigenstates of a bouncing ball

5.1 Bohr-Sommerfeld quantization

Bohr-Sommerfeld condition is

$$\oint p dz = n\hbar \tag{48}$$

Classically the energy of a ball is

$$E = \frac{p^2}{2m} + mgz \Rightarrow p = \sqrt{2m(E - mgz)} \tag{49}$$

Using this form for our momentum

$$2\sqrt{2m} \int_0^{\frac{E}{mg}} dz \sqrt{E - gmz} = 2\sqrt{mg} \frac{2}{3} \left(\frac{E}{mg} \right)^{3/2} \tag{50}$$

so

$$\frac{4}{3} \sqrt{2m} \frac{E_n^{3/2}}{mg} = n\hbar \tag{51}$$

$$E_n = \left(\frac{9}{32} mg^2 n^2 \hbar^2 \right)^{1/3} \tag{52}$$

we can use this energy to come up with an approximate maximum height for each state.

$$z_n \approx \frac{E_n}{mg} \Rightarrow z_n \approx \left(\frac{9n^2 \hbar^2}{32m^2 g} \right)^{\frac{1}{3}} \tag{53}$$

5.2 Minimization of E and the uncertainty principle

We assume the the momentum goes like $p \approx \frac{n\hbar}{z}$, which implies that energy goes like.

$$E_n \approx \frac{n^2 \hbar^2}{2mz_n^2} + mgz_n \quad (54)$$

To minimize we find the zero of the derivative

$$\partial_z E = 0 = -\frac{n^2 \hbar^2}{mz_n^3} + mg \Rightarrow z_n = \left(\frac{n^2 \hbar^2}{m^2 g} \right)^{1/3} \quad (55)$$

We can write E in terms of constants and n

$$E_n = \frac{n^2 \hbar^2}{2m} \left(\frac{m^2 g}{n^2 \hbar^2} \right)^{\frac{2}{3}} + mg \left(\frac{n^2 \hbar^2}{m^2 g} \right)^{\frac{1}{3}} \\ E_n = \frac{3}{2} (mg^2 n^2 \hbar^2)^{\frac{1}{3}} \quad (56)$$

6 Problem 6

Suppose that $|\Psi\rangle \in |\mathfrak{R}\rangle$ in position space. Then (I drop the hats here so that things look nicer).

$$\langle \Psi | \vec{p} | \Psi \rangle = -i \int d^3 r \Psi(\vec{r}) \vec{\nabla} \Psi(\vec{r}) \quad (57)$$

($\Psi = \Psi^*$) The conjugate equation is

$$\langle \Psi | \vec{p}^\dagger | \Psi \rangle = i \int d^3 r \Psi(\vec{r}) \vec{\nabla} \Psi(\vec{r}) \quad (58)$$

So

$$\langle \Psi | \vec{p} | \Psi \rangle = -\langle \Psi | \vec{p}^\dagger | \Psi \rangle \quad (59)$$

Now we note that p is a physical observable which mean that we must have REAL eigen values. Therefor \vec{p} is hermitian.

$$\langle \Psi | \vec{p} | \Psi \rangle = \langle \Psi | \vec{p}^\dagger | \Psi \rangle \quad (60)$$

which means that

$$\langle \Psi | \vec{p} | \Psi \rangle = -\langle \Psi | \vec{p} | \Psi \rangle = 0 \quad (61)$$

7 Problem 7

7.1 Wavefunction of a localized particle

At $t = 0^+$ the wavefunction is just a delta function

$$\Psi(x, 0) = A\delta(x - x_0) \quad (62)$$

(some picture here)

7.2 V

If we measure V at $t = 0^+$ then we find

$$\langle 0^+ | V(x) | 0^+ \rangle = v(x_0) \quad (63)$$

7.3 evolving the delta function

By definition

$$|\Psi(x, t)\rangle = U(t) |\Psi(x, 0)\rangle \quad (64)$$

so

$$\begin{aligned} \Psi(x, t) &= \int dx' \\ U(x, x'; t) \Psi(x', 0^+) &= U(x, x_0; t) \\ \Psi(x, t) &= A \sqrt{\frac{m}{2\pi i \hbar t}} e^{\frac{im(x-x_0)^2}{2\hbar t}} \end{aligned} \quad (65)$$

7.4 $P(t)$

Finding the Probability density as a function of time now should be really easy.

$$P(x, t) = |A|^2 \frac{m}{2\pi \hbar t} \sim O(t^{-1}) \quad (66)$$

8 Problem 8

8.1 Making sense of it all

The result in the previous problem should not make sense at all for a few reasons. First of all the $t \rightarrow 0$ limit P isn't a delta function, because there is no x -dependence. Second, $P(x, t)$ seems to be decreasing everywhere uniformly, which seems to imply that $\int P(x, t) dx = O(t^{-1})$. It is useful to look at a more physical representation of a localized state.

8.2 Finding the wavefunction at $t = 0$

$$P(x, 0) = \sqrt{\frac{1}{\pi \Delta^2}} e^{-\frac{x^2}{\Delta^2}} \quad (67)$$

which implies that

$$\Psi(x, 0) = \left(\frac{1}{\pi \Delta^2} \right)^{\frac{1}{4}} e^{-\frac{x^2}{2\Delta^2}} e^{\frac{ip_0 x}{\hbar}} \quad (68)$$

8.3 Time evolution of wavefunction

where p_0 is the momentum of the particle. Mathematically we could have just multiplied by $e^{i\phi}$, but the choice here make physical sense. To see this we look at the time evolution of the state.

$$\Psi(x, t) = \left(\frac{1}{\pi\Delta^2}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2\pi\imath a^2}} \int dx' \underbrace{e^{i\frac{(x-x')^2}{2a^2}}}_U e^{-\frac{(x')^2}{2\Delta^2} + i\frac{p_0 x'}{\hbar}} \quad (69)$$

where

$$a(t) \quad (70)$$

which we'll call the diffusion length. This is a Gaussian integral, which can be done by "completing the square." (This one comes up alot you should know how to do it). We use that

$$\int dx' e^{\imath hx' - \frac{1}{2} Ax'^2} = \sqrt{\frac{2\pi}{A}} e^{-\frac{1}{2} A^{-1} h^2} \quad (71)$$

$$\Psi(x, t) = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{\Delta + \imath \frac{a^2}{\Delta}}} e^F \quad (72)$$

where

$$F = -\frac{\left(x - \frac{p_0 a^2}{\hbar}\right)^2}{2(\Delta^2 + \imath a^2)} + \imath \frac{p_0 a^2}{\hbar} - \imath \frac{p_0^2 a^2}{2\hbar^2} \quad (73)$$

substituting in for $a^2(t) = \frac{\hbar t}{m}$.

$$\Psi = \frac{\sqrt{1 - \imath \frac{\hbar t}{m} / \Delta^2}}{(\pi\Delta^2)^{1/4} \left(1 + \frac{\hbar^2 t^2}{m^2} / \Delta^4\right)^{1/4}} e^{-\frac{\left(x - \frac{p_0 t}{m}\right)^2}{2\Delta^2 \left(1 + \frac{\hbar^2 t^2}{m^2} / \Delta^4\right)}} e^{\frac{\imath}{2} \frac{\hbar t / m}{\Delta^4 + \hbar^2 t^2 / m^2} \left(x - \frac{p_0}{m} t\right)^2 + \imath \frac{p_0 x}{\hbar} - \imath \frac{p_0^2 t}{2m\hbar}} \quad (74)$$

8.4 The Probability density

So the Probability density is

$$\begin{aligned} P(x, t) &= |\Psi(x, t)|^2 \\ &= \frac{1}{\sqrt{(\pi\Delta^2) \left(1 + \frac{\hbar^2 t^2}{m^2} / \Delta^4\right)}} e^{-\frac{\left(x - \frac{p_0 t}{m}\right)^2}{\Delta^2 \left(1 + \frac{\hbar^2 t^2}{m^2} / \Delta^4\right)}} \end{aligned} \quad (75)$$

8.5 Expectation values

8.5.1 x

The expectation value of $x(t)$ gives a nice result.

$$\langle t|x|t \rangle = \left(\frac{1}{\pi^{1/2}\Delta(t)} \right) \int dx \left[\left(x - \frac{p_0 t}{m} \right) + \frac{p_0 t}{m} \right] e^{-\frac{(x - \frac{p_0 t}{m})^2}{\Delta(t)^2}} \quad (76)$$

where

$$\Delta(t) = \Delta \sqrt{1 + \frac{\hbar^2 t^2}{m^2} / \Delta^4} \quad (77)$$

Writing $x = x - \frac{p_0 t}{m} + \frac{p_0 t}{m}$ make this integral trivial.

$$\langle x(t) \rangle = \frac{p_0 t}{m} \quad (78)$$

8.5.2 x^2

To calculate the variance we'll need to know

$$\langle x^2 \rangle = \sqrt{\frac{1}{\pi\Delta(t)^2}} \int dx (x - x_0 + x_0)^2 e^{-\frac{(x-x_0)^2}{\Delta(t)^2}} \quad (79)$$

$$= x_0(t)^2 + \frac{1}{\sqrt{\pi\Delta(t)^2}} \int dy y^2 e^{-\frac{y^2}{\Delta(t)^2}} \quad (80)$$

here we call $y = x - x_0(t)$. This leads to

$$\langle x^2 \rangle = x_0^2(t) + \frac{\Delta(t)^2}{2} \quad (81)$$

8.5.3 p

we can calculate the momentum

$$\begin{aligned} \langle p(t) \rangle &= \frac{1}{\sqrt{\pi\Delta^2(t)}} \int dx e^{-\frac{(x-x_0(t))^2}{2(\Delta^2 - ia^2)}} e^{-i\frac{p_0 x}{\hbar} + i\frac{p_0^2 t}{2m\hbar}} \\ &\quad - i\hbar \partial_x \left(e^{-\frac{(x-x_0(t))^2}{2(\Delta^2 + ia^2)}} e^{i\frac{p_0 x}{\hbar} - i\frac{p_0^2 t}{2m\hbar}} \right) \end{aligned} \quad (82)$$

The derivative above requires a product rule. One term has the form $\int dx x e^{-ax^2} = 0$. So

$$\langle p(t) \rangle = \frac{p_0}{\sqrt{\pi\Delta(t)^2}} \int dx e^{-\frac{(x-x_0(t))^2}{\Delta^2(t)}} \quad (83)$$

$$\langle t|p|t \rangle = p_0 \quad (84)$$

8.5.4 p^2

we would also like to know

$$\begin{aligned}
\langle p(t)^2 \rangle &= -\frac{\hbar^2}{\sqrt{\pi\Delta^2(t)}} \int dx e^{-\frac{(x-x_0(t))^2}{2(\Delta^2-ia^2)}} e^{-i\frac{p_0x}{\hbar} + i\frac{p_0^2t}{2m\hbar}} \\
&\quad \partial_x^2 \left(e^{-\frac{(x-x_0(t))^2}{2(\Delta^2+ia^2)}} e^{i\frac{p_0x}{\hbar} - i\frac{p_0^2t}{2m\hbar}} \right) \\
&= -\frac{\hbar^2}{\sqrt{\pi\Delta^2(t)}} \int dx e^{-\frac{(x-x_0(t))^2}{2(\Delta^2-ia^2)}} e^{-i\frac{p_0x}{\hbar} + i\frac{p_0^2t}{2m\hbar}} \\
&\quad \partial_x \left[\left(-\frac{x-x_0}{\Delta^2+ia^2} + i\frac{p_0}{\hbar} \right) e^{-\frac{(x-x_0(t))^2}{2(\Delta^2+ia^2)}} \right. \\
&\quad \left. e^{i\frac{p_0x}{\hbar} - i\frac{p_0^2t}{2m\hbar}} \right] \\
&= -\frac{\hbar^2}{\sqrt{\pi\Delta^2(t)}} \int dx e^{-\frac{(x-x_0(t))^2}{2(\Delta^2-ia^2)}} \\
&\quad e^{-i\frac{p_0x}{\hbar} + i\frac{p_0^2t}{2m\hbar}} \left[-\frac{1}{\Delta^2+ia^2} + \left(\frac{x-x_0}{\Delta^2+ia^2} \right)^2 - \frac{p_0^2}{\hbar^2} - \frac{2ip_0(x-x_0)}{\hbar(\Delta^2+ia^2)} \right] \\
&\quad e^{-\frac{(x-x_0(t))^2}{2(\Delta^2+ia^2)}} e^{i\frac{p_0x}{\hbar} - i\frac{p_0^2t}{2m\hbar}} \\
&= \frac{1}{\sqrt{\pi\Delta^2(t)}} \int dx \left[\frac{\hbar^2}{\Delta^2+ia^2} - \left(\frac{\hbar}{\Delta^2+ia^2} \right)^2 (x-x_0)^2 + p_0^2 + \right. \\
&\quad \left. \underbrace{i\frac{2p_0\hbar(x-x_0)}{\Delta^2+ia^2}}_{f=0 \text{ by sym}} \right] e^{\frac{\Delta^2(x-x_0)^2}{\Delta^4+a^4}} \\
&= p_0^2 + \frac{\hbar^2}{\Delta^2+ia} \left(1 - \frac{1}{\Delta^2+ia^2} \frac{\Delta^4+a^4}{2\Delta^2} \right) \\
&= p_0^2 + \frac{\hbar^2}{\Delta^2+ia} \underbrace{\left(1 + \frac{\Delta^2-ia^2}{2\Delta^2} \right)}_{(\Delta^2+ia^2)/(2\Delta^2)} \\
\langle tp^2|t \rangle &= p_0^2 + \frac{\hbar^2}{2\Delta^2} \tag{85}
\end{aligned}$$

the RMS uncertainty on each p and x is

$$\delta p_{rms} = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar}{\sqrt{2}\Delta} \tag{86}$$

$$\delta x_{rms} = \frac{\Delta(t)}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Delta \sqrt{1 + \frac{\hbar^2 t^2}{m^2 \Delta^4}} \tag{87}$$

and so

$$\delta x_{rms} \delta p_{rms} = \frac{\hbar}{2} \sqrt{1 + \frac{\hbar^2 t^2}{m^2}} / \Delta^4 \quad (88)$$

9 Problem 9

Proving some standard commutation relations

9.1 a

$$\begin{aligned} [A, BC] &= ABC - BCA \\ &= ABC - BAC - BCA + BAC \\ [A, BC] &= [A, B]C + B[A, C] \end{aligned} \quad (89)$$

9.2 b

We look at this in the context of Taylor expansion. First look at

$$\underbrace{[p, x^n]}_{f_n} = \underbrace{[p, x^{n-1}]}_{f_{n-1}} x + x^{n-1} \underbrace{[p, x]}_{-i\hbar=f_1} \quad (90)$$

This gives us the relation

$$f_n = f_{n-1} x + x^{n-1} (-i\hbar) \quad (91)$$

by iterating this expression n -times.

$$\begin{aligned} [p, x^n] &= nx^{n-1} [p, x] \\ [p, x^n] &= -in x^{n-1} \end{aligned}$$

which is the derivative of x^n . So via Taylor series

$$[p, f(x)] = -i\hbar f'(x) \quad (92)$$

9.3 what is $[e^{i\frac{pa}{\hbar}}, x]$

This is easiest to show in p-space, where $x_p = i\hbar\partial_p$. So

$$\begin{aligned} \left[e^{i\frac{pa}{\hbar}}, i\hbar\partial_p \right] \psi(p) &= (e^{i\frac{pa}{\hbar}} i\hbar\partial_p - i\hbar\partial_p e^{i\frac{pa}{\hbar}}) \psi(p) \\ &= e^{i\frac{pa}{\hbar}} i\hbar\partial_p \psi(p) + a e^{i\frac{pa}{\hbar}} \psi(p) - e^{i\frac{pa}{\hbar}} i\hbar\partial_p \psi(p), \\ &= a e^{i\frac{pa}{\hbar}} \end{aligned} \quad (93)$$

We can also use an argument similar to 8.2.

9.4 Show that T_a is the translation operator

suppose

$$T_a = e^{i\frac{pa}{\hbar}} \quad (94)$$

then

$$T_a f(x) T_a^\dagger = f(x) + [T_a, f(x)] T_a^\dagger \quad (95)$$

Once again in the context of Taylor expansion we look at

$$T_a x^n T_a^\dagger \quad (96)$$

for $n = 1$ equation (97) becomes

$$T_a x T_a^\dagger = x + [T_a, x] T_a^\dagger \quad (97)$$

This commutator is (94) so,

$$T_a f(x) T_a^\dagger = f(x + a) \quad (98)$$

For $n \neq 1$ we note that T_a is unitary (ie $T_a^\dagger T_a = I$), so

$$T_a x^n T_a^\dagger = T_a x T_a^\dagger T_a x T_a^\dagger \cdots T_a x T_a^\dagger = (x + a)^n \quad (99)$$

So,

$$T_a f(x) T_a^\dagger = f(x + a) \quad (100)$$

This can also be done in p -space.

$$f(x, a) \equiv e^{i\frac{pa}{\hbar}} f(x) e^{-i\frac{pa}{\hbar}} \quad (101)$$

$$\partial_a f(x, a) = \frac{i}{\hbar} [p, f(x, a)] = \partial_x f(x, a) \quad (102)$$

A solution to this equation is

$$f(x, a) = e^{a\partial_x} f(x) \quad (103)$$

which is just the Taylor expansion of $f(x + a)$ around $f(x)$; therefore,

$$f(x, a) = f(x + a) \quad (104)$$

10 problem 10

10.1 show p is hermitian in x -space

$$p \equiv -i\hbar\partial_x \quad (105)$$

hermiticity is defined as

$$\langle A \rangle = \langle A^\dagger \rangle \quad (106)$$

I'll just calculate both and show that they are equal. Hermite conjugate operators act backwards.

$$\begin{aligned}
 \langle p \rangle &= \int \Psi^* (-i\hbar \partial_x \Psi) \\
 \langle p^\dagger \rangle &= \int i\hbar (\partial_x \Psi^*) \Psi \\
 \langle p^\dagger \rangle &= - \int \Psi^* i\hbar \partial_x \Psi + \underbrace{\Psi^* \Psi}_{=0} \Big|_{-\infty}^{\infty} \\
 \langle p^\dagger \rangle &= \langle p \rangle
 \end{aligned} \tag{107}$$

10.2 A hermitian x, p product

px is not hermitian because

$$(px)^\dagger = x^\dagger p^\dagger = xp = px + [x, p] \neq px \tag{108}$$

but

$$\frac{1}{2} (px + xp)^\dagger = \frac{1}{2} (xp + px) = \frac{1}{2} (px + xp) \tag{109}$$

is Hermitian.