Quantum Mechanics - 1: HW 3 Solutions

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1 Problem 1

Consider a particle in the ground state of a box of length a. The wave-function of this state is

$$\psi_0(x,0) = \begin{cases} \sqrt{\frac{2}{a}}\cos\left(\frac{\pi}{a}x\right) & x \in \left[-\frac{a}{2}, \frac{a}{2}\right] \\ 0 & \text{else} \end{cases} \tag{1}$$

1.1 Momentum Distribution

To get the momentum distribution we need to decompose the positions states into momentum eigen state $\phi_k(x) = e^{\imath kx}$. We find the coefficient for each term by taking the Fourier transform.

$$||\psi_0(x)\rangle = \int |k\rangle \langle k|\psi_0(x)\rangle \, dk \tag{2}$$

$$\begin{aligned}
\langle k|\psi_1\rangle &= \sqrt{\frac{2}{a}}\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(\frac{\pi}{a}x\right) e^{-\imath kx} \\
&= \frac{1}{2}\sqrt{\frac{2}{a}}\int_{-\frac{a}{2}}^{\frac{a}{2}} \left[e^{\imath\left(\frac{\pi}{a}-k\right)x} + e^{-\imath\left(\frac{\pi}{a}+k\right)x}\right] \\
&= \frac{1}{2}\sqrt{\frac{2}{a}}\left[\frac{1}{\imath\left(\frac{\pi}{a}-k\right)}\left(e^{\imath\left(\frac{\pi}{a}-k\right)\frac{a}{2}} - e^{-\imath\left(\frac{\pi}{a}-k\right)\frac{a}{2}}\right) + \frac{1}{-\imath\left(\frac{\pi}{a}+k\right)}\left(e^{-\imath\left(\frac{\pi}{a}+k\right)\frac{a}{2}} - e^{\imath\left(\frac{\pi}{a}+k\right)\frac{a}{2}}\right)\right] \\
&= \sqrt{\frac{2}{a}}\left[\frac{1}{\frac{\pi}{a}-k}\cos\left(\frac{ka}{2}\right) + \frac{1}{\frac{\pi}{a}+k}\cos\left(\frac{ka}{2}\right)\right] \\
\langle k|\psi_0\rangle &= \sqrt{\frac{2}{a}}\frac{2\pi/a}{\left(\frac{\pi}{a}\right)^2 - k^2}\cos\left(\frac{ka}{2}\right) = \tilde{\Psi}(k,0)
\end{aligned} \tag{3}$$

The ϕ_k's are normalized and orthogonal, so the Probability is just $|\langle k|\psi_0\rangle|^2$

$$\frac{8\pi^2}{a^3}\frac{\cos^2\left(\frac{ka}{2}\right)}{\left(\left(\frac{\pi}{a}\right)^2 - k^2\right)^2} \tag{4}$$

1.2 adiabatic expansion

Suppose the potential is shut off suddenly then each momentum eigen state will evolve with a factor of $e^{-\imath \frac{p^2}{2m\hbar} t}$. So our time dependent wave-function function will just be the sum of these.

$$\begin{aligned}\Psi(x,t) &= \int \frac{dk}{2\pi} \tilde{\Psi}(k,0)\, e^{\imath kx} e^{-\imath \frac{\hbar^2 k^2}{2m} t/\hbar} \\ \Psi(x,t) &= \int \frac{dk}{2\pi} \sqrt{\frac{2}{a}} \frac{2\pi}{a} \frac{\cos\left(\frac{ka}{2}\right)\cos(kx)}{\left(\frac{\pi}{a}\right)^2 - k^2} e^{-\imath \frac{\hbar k^2}{2m} t}\end{aligned} \tag{5}$$

In this last part we used parity to argue that inside the integral we can replace $e^{\imath kx}$ by $\cos kx$ and not change the result.

1.3 ground state after adiabatic expansion

Consider the case where the length of the box is doubled very quickly. The probability that the we end up in the new ground state is just.

$$|\langle \Psi_0' | \Psi_0 \rangle|^2 \tag{6}$$

Where

$$\langle x | \Psi' \rangle = \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{2a}\right) \tag{7}$$

so,

$$c_1 = \langle \Psi_0' | \Psi_0 \rangle = \frac{2 \times 2}{a\sqrt{2}} \int_0^{\frac{a}{2}} \cos\left(\frac{\pi}{2a} x\right) \cos\left(\frac{\pi}{a} x\right) = \frac{8}{3\pi} \tag{8}$$

therefore

$$P_{0->0'} = \left(\frac{8}{3\pi}\right)^2 \tag{9}$$

2 Problem 2

2.1 Show that $\langle \psi | H | \psi \rangle \geq E_0$

This important result can be proved by decomposing our general state into energy eigen states of the Hamiltonian.

$$|\Psi\rangle = \sum C_n |E_n\rangle \tag{10}$$

Because this state is normalized $\sum |C_n|^2 = 1$. Also note that $\langle E_n | E_{n'} \rangle = \delta_{nn'}$. The second statement means that we can write the energy of our state

$$\begin{aligned}\langle \Psi | H | \Psi \rangle &= \sum |C_n|^2 E_n \\ &= E_0 + \sum |C_n|^2 \epsilon_n \\ &\geq E_0\end{aligned} \tag{11}$$

where $\epsilon_n \equiv E_n - E_0 \geq 0$.

The above is an important result which can be used as a variation principle to find approximate ground state wave-functions.

2.2 Attractive potentials have at least on ground state

Suppose we have the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - |V(x)| \tag{12}$$

We will use the result from part 2.1 by choosing a trial wave-function

$$\psi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha x^2}{2}} \tag{13}$$

and varying α. Our action is just the energy

$$\begin{aligned}
\delta_\alpha E(\alpha) &= \delta_\alpha \langle \psi_\alpha | H | \psi_\alpha \rangle = 0 \\
&= \frac{\partial}{\partial \alpha} \int dx \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} \left[\left(\alpha - \alpha^2 x^2\right) \frac{\hbar^2}{2m} - |V(x)| \right] \\
&= \frac{\partial}{\partial \alpha} \left[\left(\frac{\alpha}{2}\right) \frac{\hbar^2}{2m} - \int dx \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} |V(x)| \right] \\
&= \frac{\hbar^2 \alpha}{4m} - \int dx \left[\frac{1}{2}\sqrt{\frac{1}{\alpha \pi}} - \sqrt{\frac{\alpha}{\pi}} x^2 \right] |V(x)| \, e^{-\alpha x^2}
\end{aligned} \tag{14}$$

To show that the minimum here is negative we'll look at the limits of $E(\alpha)$ and $E'(\alpha)$. To make the $\alpha \to \infty$ limit of $E(\alpha)$ easier to see, I will rewrite E in terms of $z = \sqrt{\alpha} x$

$$E(\alpha) = \frac{\hbar^2 \alpha}{4m} - \frac{1}{\sqrt{\pi}} \int_{\infty}^{\infty} dz e^{-z^2} \left| V\left(\frac{z}{\sqrt{a}}\right) \right| \tag{15}$$

As $\alpha \to \infty$ the first term goes as α, but the integral approaches the constant $\sqrt{\pi} V(0)$. (draw a picture). So

$$\lim_{\alpha \to \infty} E(\alpha) = \frac{\hbar^2}{4m} \alpha \tag{16}$$

The $\alpha \to 0$ limit of E is obviously zero. The $\alpha \to 0$ of $\partial_\alpha E$ is dominated by the integral term, because the $-\alpha^{-1/2}$ factor diverges to $-\infty$. Which means that $E(\alpha) < 0$ for small α, and it must turn around at some point to match our large α limit. Therefore,

$$0 > E_{min}(\alpha > 0) > E_0 \Rightarrow \text{ground state is a bound solution} \tag{17}$$

3 Problem 3

Consider the Potential $V(x) = -U_0 \delta(x)$.

3.1 finding eigenstate with $E = -\frac{mU_0^2}{2\hbar^2}$

Because our potential is symmetric we expect only even and odd solutions. Odd solutions are always 0 at the origin, and thus will not see the potential at all (ie. these will all be free states). We will consider only even solutions, which have the form

$$\psi = \begin{cases} e^{-\kappa x} & x > 0 \\ e^{\kappa x} & x < 0 \end{cases} \tag{18}$$

solving the wave equation

$$-\frac{\hbar^2}{2m}\psi'' - U_0\delta\left(x\right)\psi = E\psi \tag{19}$$

away from $x = 0$. We find that

$$E = -\frac{\hbar^2\kappa^2}{2m} \tag{20}$$

We find the restriction of κ by integrating the wave equation.

$$\int_{-\epsilon}^{\epsilon}\left[-\frac{\hbar^2}{2m}\psi'' - U_0\delta\left(x\right)\psi = E\psi\right] \tag{21}$$

$$-\frac{\hbar^2}{2m}\left(\psi'\left(0^+\right) - \psi'\left(0^-\right)\right) - U_0\psi\left(0\right) = 2E\epsilon\psi\left(0\right) \tag{22}$$

$$\frac{\hbar^2\kappa}{m} = U_0 \Rightarrow \kappa = \frac{mU_0}{\hbar^2} \tag{23}$$

There is only one bound state with energy.

$$E = -\frac{\hbar^2\kappa^2}{2m} = -\frac{mU_0^2}{2\hbar^2} \tag{24}$$

3.2 Free State Spectrum

For odd states

$$\psi_k^{(odd)}\left(x\right) = \sin kx, \text{ where } E = \frac{\hbar^2k^2}{2m} \tag{25}$$

These states have a continuous spectrum. For even states

$$\psi_k^{(even)}\left(x\right) = \begin{cases} \cos\left(kx+\delta\right) & x > 0 \\ \cos\left(kx-\delta\right) & x < 0 \end{cases} \tag{26}$$

Integrating the wave equation like before

$$-\frac{\hbar^2}{2m}\left(\psi'\left(0^+\right) - \psi\left(0^-\right)\right) = u_0\cos\delta \tag{27}$$

we get the restriction on k

$$\frac{\hbar k}{mU_0} = \cot \delta_k \tag{28}$$

δ_k is arbitrary, so we still have a continuous energy spectrum $E = \frac{\hbar^2 k^2}{2m}$. The first derivative of the even wave functions are discontinuous at $x = 0$ and we have a phase shift δ_k.

3.3 Positive δ

3.3.1 No bound state

If $U_0 \to -U_0$, then $\kappa < 0$, which gives us $e^{|\kappa x|}$ solutions. These don't make sense, so there are no bound solutions.

3.3.2 new phase shift

As before odd states don't change

$$\psi_k^{(odd)}(x) = \sin kx \tag{29}$$

Even solutions are shifted away from the origin instead of toward the origin

$$\psi_k^{(even)}(x) = \begin{cases} \cos(kx + \delta_k) & x > 0 \\ \cos(kx - \delta_k) & x < 0 \end{cases} \tag{30}$$

where

$$\cot \delta_k = -\frac{\hbar^2 k}{mU_0} \tag{31}$$

4 Problem 4

Consider the potential

$$V(x) = \begin{cases} 0 & |x| \le a \\ V_0 & |x| > a \end{cases} \tag{32}$$

4.1 The Parity Operator

Consider the parity operator P.

$$P\psi(x) = \psi(-x) \tag{33}$$

and the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \tag{34}$$

The commutator

$$\begin{aligned}
[P,H]\,\psi &= (PH - HP)\,\psi \\
&= H\,(-x)\,\psi\,(-x) - H\psi\,(-x) \\
&= H\,(x)\,\psi(-x) - H\,(x)\,\psi(-x), \text{ because } V\,(-x) = V\,(x) \\
[P,H] &= 0
\end{aligned} \tag{35}$$

suppose ψ_E is an energy eigenfunction. Then

$$H\psi_E = E\psi_E \tag{36}$$

$$\begin{aligned}
PH\psi_E &= EP\psi_E \\
HP\psi_E &= EP\psi_E \text{ using the commutator} \\
H\,(P\psi_E) &= E\,(P\psi_E)
\end{aligned} \tag{37}$$

There is no degeneracy in the 1-d case so this state must be $c\psi_E$.

$$P^2 = I \tag{38}$$

so

$$\lambda_p = \pm 1 \Rightarrow c = \pm 1 \tag{39}$$

$$P\psi_E^{\pm} = \pm\psi_E^{\pm} \tag{40}$$

where ψ_E^+ is even and ψ_E^- is odd.

4.2 Energy Spectrum

We'll match the BC at $x = a$. Either side should give the same result.

4.2.1 Even solutions

the general form of the even bound solutions is

$$\psi_{even}\,(x) = \begin{cases} Ae^{\kappa x} & x < -a \\ B\cos kx & -a < x < a \\ Ae^{-\kappa x} & x > a \end{cases} \tag{41}$$

We match both the wavefucntion and it's first derivative at the boundary.

$$B\cos ka = Ae^{-\kappa a} \tag{42}$$

$$-Bk\sin ka = -A\kappa e^{-\kappa a} \tag{43}$$

these give us the relation

$$k\tan ka = \kappa \tag{44}$$

and

$$\frac{A}{B} = e^{\kappa a}\cos ka \tag{45}$$

4.2.2 Odd Solution

$$\psi_{odd}(x) = \begin{cases} -Ce^{\kappa x} & x < -a \\ D\sin ka & -a < x < a \\ Ce^{-\kappa x} & x > a \end{cases} \tag{46}$$

matching wave-function and it's first derivative.

$$D\sin ka = Ce^{-\kappa a} \tag{47}$$

$$Dk\cos ka = -C\kappa e^{-\kappa a} \tag{48}$$

This leads to the relation

$$k\cot ka = -\kappa \tag{49}$$

note that

$$\frac{\hbar^2 k^2}{2m} = E \tag{50}$$

$$\frac{\hbar^2 \kappa^2}{2m} = V_0 - E \tag{51}$$

so

$$k^2 + \kappa^2 = \frac{2mV_0}{\hbar^2} \tag{52}$$

4.3 Graphical solution for energy spectrum

To find the energies of our particle in a box we look at the intersection of

$$\beta = \alpha\tan\alpha \tag{53}$$

$$\beta^2 + \alpha^2 = \frac{2mV_0a^2}{\hbar^2} \tag{54}$$

for even solutions and

$$\beta = -\alpha\cot\alpha \tag{55}$$

$$\beta^2 + \alpha^2 = \frac{2mV_0a^2}{\hbar^2} \tag{56}$$

for odd solutions

Note there is always an even bound state, but if

$$\frac{2mV_0a^2}{\hbar^2} < \frac{\pi}{2} \tag{57}$$

Then there will be no odd bound solutions. On the figure there is only one even solutions for $V_0 = V_{01}$, and three solutions (two even and one odd) for $V_0 = V_{02}$.

4.4 A sketch of low energy solutions

The three lowest energy solutions are plotted below.

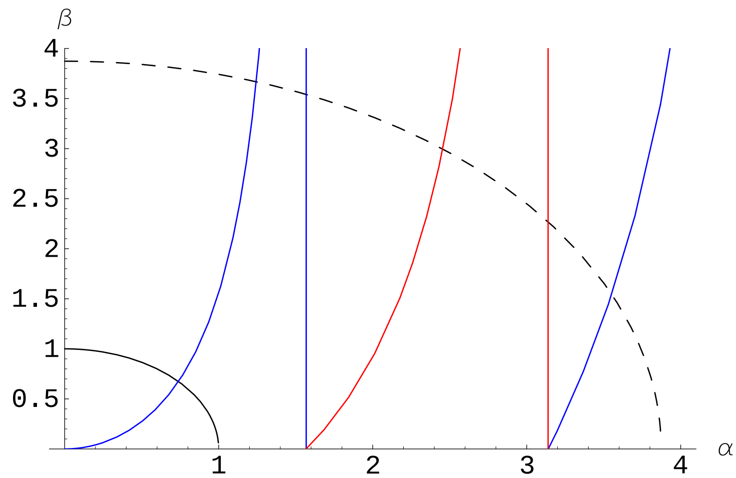

Figure 1: Blue/red line intersection give us even/odd-solutions. In units of $\frac{2ma^2}{\hbar^2}$

4.5 Always an even solution

As I said before there is only one even solution if

$$V_0 < \frac{\hbar^2\pi^2}{8ma^2} \tag{58}$$

We can see this by looking at figure 1.

4.6 A ground state wave-function

Here we calculate the ground state wave-function for $V_0 \frac{\hbar^2\pi^2}{8ma^2}$. We know it will be an even wave-function.

$$\psi_0\left(x\right) = \begin{cases} Ae^{\kappa x} & x < -a \\ B\cos kx & 0 \geq x \leq a \\ Ae^{-\kappa x} & x > a \end{cases} \tag{59}$$

Numerically solving for (56), we find

$$\alpha_0 = 0.934 \tag{60}$$

$$\beta_0 = 1.263 \tag{61}$$

$$\frac{A}{B} = 2.102 \tag{62}$$

A is found by normalization of our state.

$$1 = 2\int_0^\infty \left|\psi_{gs}\right|^2 dx \tag{63}$$

$$
\begin{aligned}
\frac{1}{2} &= \int_0^a dx B^2 \cos kx + \int_0^\infty dx A^2 e^{-2\kappa x} \\
&= \frac{B^2}{k}\left(\frac{1}{2}\alpha_0 + \frac{1}{4}\sin 2\alpha_0\right) + \frac{A^2}{\kappa}\frac{1}{2}e^{-2\beta_0} \\
&= aB^2\left[\frac{1}{ak}\left(\frac{1}{2}\alpha_0 + \frac{1}{4}\sin 2\alpha_0\right) + \underbrace{\frac{A^2}{B^2}}_{e^{2\beta_0}\cos^2\alpha_0}\frac{1}{2\kappa a}e^{-2\beta_0}\right] \\
&= \frac{1}{\sqrt{a}}\left[1 + \frac{1}{2\alpha_0}\sin 2\alpha_0 + \frac{1}{\beta_0}\cos^2\alpha_0\right]^{-\frac{1}{2}}
\end{aligned}
$$

so

$$B = \frac{0.747}{\sqrt{a}} \approx \frac{\pi}{4\sqrt{a}} \tag{64}$$

$$A = \frac{1.570}{\sqrt{a}} \approx \frac{\pi}{2\sqrt{a}} \tag{65}$$

4.7 Infinite square well limit

In the infinite square well limit $\kappa \to \infty$. We will treat even and odd solutions separately.

4.7.1 even

$$k \tan ka = \kappa \tag{66}$$

$$k = \frac{\pi n}{2a},\ n \in odd \tag{67}$$

4.7.2 odd

$$k \cot ka = -\kappa \tag{68}$$

$$k = \frac{\pi n}{2a},\ n \in even \tag{69}$$

4.7.3 Together

$$E_k = \frac{n^2\pi^2 n^2}{8ma^2},\ \ n \in \mathbb{Z} \tag{70}$$

Which is the infinite square well result.

4.8 Continuous limit

4.8.1 even

$$\psi(x) = \begin{cases} \cos kx & |x| \le a \\ C \sin k'x + D \cos k'x = A \cos(k'x + \delta) & |x| > a \end{cases} \tag{71}$$

where

$$\begin{aligned} k &= \frac{\sqrt{2mE}}{\hbar} \\ k' &= \frac{\sqrt{2m\left(E-V_0\right)}}{\hbar} \end{aligned} \tag{72}$$

we can relate the coefs. via B.C.'s

$$\begin{cases} \cos ka = C\sin k'a + D\cos k'a \\ -k\sin ka = Ck'\cos k'a - Dk'\sin k'a \end{cases} \tag{73}$$

Solve as matrix equation

$$\begin{pmatrix} \sin k'a & \cos k'a \\ k'\cos k'a & -k'\sin k'a \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \cos ka \\ -k\sin ka \end{pmatrix} \tag{74}$$

we find the inverse

$$M^{-1} = \begin{pmatrix} \sin k'a & \frac{1}{k'}\cos k'a \\ \cos k'a & -\frac{1}{k'}\sin k'a \end{pmatrix} \tag{75}$$

We find

$$C^{(even)} = \cos ka \sin k'a - \frac{k}{k'}\sin ka \cos k'a \tag{76}$$

$$D^{(even)} = \cos ka \cos k'a + \frac{k}{k'}\sin ka \sin k'a \tag{77}$$

$$\cot\delta \equiv -\frac{D^{(even)}}{C^{(even)}} \tag{78}$$

4.8.2 odd

$$\psi\left(x\right) = \begin{cases} \sin kx & |x| \le a \\ C\sin k'x + D\cos k'x = A\cos\left(k'x+\delta\right) & |x| > a \end{cases} \tag{79}$$

where once again

$$\begin{aligned} k &= \frac{\sqrt{2mE}}{\hbar} \\ k' &= \frac{\sqrt{2m\left(E-V_0\right)}}{\hbar} \end{aligned} \tag{80}$$

we can relate the coefs. via B.C.'s

$$\begin{cases} \sin ka = C\sin k'a + D\cos k'a \\ k\cos ka = Ck'\cos k'a - Dk'\sin k'a \end{cases} \tag{81}$$

Solve as matrix equation

$$\begin{pmatrix} \sin k'a & \cos k'a \\ k'\cos k'a & -k'\sin k'a \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \sin ka \\ -k\cos ka \end{pmatrix} \tag{82}$$

We find

$$C^{(odd)} = \sin ka \sin k'a + \frac{k}{k'} \cos ka \cos k'a \tag{83}$$

$$D^{(odd)} = \sin ka \cos k'a + \frac{k}{k'a} - \frac{k}{k'} \cos ka \sin k'a \tag{84}$$

$$\tan \delta = \frac{D^{(odd)}}{C^{(odd)}} \tag{85}$$

4.9 Reducing to δ

First we shift the energy so that $V_{outside} = 0$, so that $\kappa = \frac{\sqrt{2mE}}{\hbar^2}$. In this limit $ka \to 0$. We treat bound and unbound cases separately.

4.9.1 bound

$$k \tan ka \to k^2 a = \kappa \tag{86}$$

$$\left(\frac{2mV_0}{\hbar^2}\right) a = \kappa \tag{87}$$

$$\kappa = \frac{mU_0}{\hbar^2} \tag{88}$$

$$E_{bound} = -\frac{mU_0^2}{2\hbar^2} \tag{89}$$

For our normalization factors

$$\frac{A}{B} = e^{\kappa a} \cos ka \to 1 \tag{90}$$

$$\begin{aligned} 1 &= \int |\psi|^2 \, dx \qquad (91) \\ &= 2A^2 \int_0^\infty e^{-2\kappa x} dx \\ &= \frac{A^2}{\kappa} \end{aligned}$$

(92)

$$A = \sqrt{\kappa} \tag{93}$$

4.9.2 unbound

from part h. in the limit $ka \sim 1/\sqrt{a} \to 0$. For even

$$D^{(even)} \to 1 + \frac{k^2 a}{k'} k'a \approx 1 \tag{94}$$

$$\cot \delta_{even} = -\frac{D^{(even)}}{C^{(even)}} \to \frac{\hbar^2 k'}{mU_0} \tag{95}$$

(96)

For odd

$$C^{(odd)} \to \infty \tag{97}$$

$$D^{(odd)} \to 0 \tag{98}$$

$$\frac{C^{(odd)}}{D^{(odd)}} = \cot\delta_0 \to \infty \Rightarrow \delta = 0 \tag{99}$$

This limit reduces to the δ case as expected.

5 Problem 5

Compute

$$j = -\frac{\imath\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{100}$$

for

$$\psi = Ae^{\imath px/\hbar} + Be^{-\imath px/\hbar} \tag{101}$$

$$\nabla\psi = \frac{\imath p}{\hbar}\left(Ae^{-\imath px/\hbar} - Be^{\imath px/\hbar}\right) \tag{102}$$

so

$$\begin{aligned} j &= -\frac{\imath\hbar}{2m}\frac{\imath p}{\hbar}\left[\left(A^* e^{-\imath px/\hbar} + B^* e^{\imath px/\hbar}\right)\left(Ae^{\imath px/\hbar} - Be^{-\imath px/\hbar}\right) + C.C.\right] \\ &= \frac{p}{2m}\left[|A|^2 - |B|^2 + AB^* e^{2\imath px/\hbar} - A^* B e^{-2\imath px/\hbar} + C.C\right] \end{aligned} \tag{103}$$

$$j = \frac{p}{m}\left[|A|^2 - |B|^2\right] \tag{104}$$

6 Problem 6

6.1 Wave packet Oscillations

consider the wave-function

$$\psi(x,0) = \left(\frac{1}{\pi x_0^2}\right)^{\frac{1}{4}} e^{-\frac{(x-a)^2}{2x_0^2}} \tag{105}$$

6.1.1 time evolution

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} C_n |E_n\rangle e^{-\frac{\imath}{\hbar}E_n t} \tag{106}$$

where

$$C_n = \langle E_n|\psi(0)\rangle \tag{107}$$

$$\langle x|\psi(t)\rangle = \sum_{n=0}^{\infty} C_n \langle x|E_n\rangle e^{-\frac{-\imath}{\hbar}E_n t} \tag{108}$$

$$\psi(x,t) = \sum_{n=0}^{\infty} C_n \phi_n(x) e^{-\frac{\imath}{\hbar}E_n t} \tag{109}$$

where

$$\phi_n(x) = N_n H_n(x/x_0) e^{-\frac{x^2}{2x_0^2}},\ N_n = \left(\frac{1}{x_0 \pi^{1/2} 2^n n!}\right) \tag{110}$$

is an energy eigen state

6.1.2 The coefficients

$$C_n = \langle E_n|\psi(0)\rangle \tag{111}$$

$$= \int dx \psi(x,0) \phi_n^*(x) \tag{112}$$

$$= \int dx N_n \left(\frac{1}{\pi x_0^2}\right)^{\frac{1}{4}} e^{-\frac{(x-a)^2}{2x_0^2}} H_n\left(\frac{x}{x_0}\right) e^{-\frac{x^2}{2x_0^2}} \tag{113}$$

$$= \frac{N_n x_0}{(\pi x_0^2)^{1/4}} \int dx e^{-\frac{(x-a)^2}{2}} H_n(x) e^{-\frac{x^2}{2}} \tag{114}$$

We can take this integral with the help of the generating function

$$Z(x,s) = e^{-s^2+2sx} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} s^n \tag{115}$$

$$I(s) = \int dx e^{-\frac{(x-a)^2}{2}} e^{-S^2+2Sx} e^{-\frac{x^2}{2}} \tag{116}$$

$$I(S) = \int dx e^{-x^2+ax+2Sx} e^{-S^2-\frac{1}{2}a^2} \tag{117}$$

$$= e^{-S^2-\frac{1}{2}a^2} \sqrt{\pi} e^{\frac{1}{4}(a+2s)^2} \tag{118}$$

$$= \sqrt{\pi} e^{-\frac{1}{4}a^2+aS} \tag{119}$$

$$= \sqrt{\pi} e^{-\frac{1}{4}a^2} \sum_{n=0}^{\infty} \frac{a^n}{n!} S^n \tag{120}$$

On the other hand

$$I(S) = \int dx e^{-\frac{(x-a)^2}{2}} \sum_n \frac{H_n(x)}{n!} S^n e^{-\frac{x^2}{2}} \tag{121}$$

$$= \sum_n \int dx e^{-\frac{(x-a)^2}{2}} H_n(x) e^{-\frac{x^2}{2}} \frac{S^n}{n!} \tag{122}$$

This means that

$$\int dx e^{-\frac{(x-a)^2}{2}} H_n(x) e^{-\frac{x^2}{2}} = \sqrt{\pi} a^n e^{-\frac{1}{4}a^2} \tag{123}$$

so

$$C_n = \sqrt{\frac{\pi}{x_0 \pi 2^n n!}} \frac{1}{\pi^{1/4} x_0^{1/2}} x_0 a^n e^{-\frac{1}{4}a^2} \tag{124}$$

$$C_n = \frac{a^n e^{-\frac{1}{4}a^2}}{\sqrt{2^n n!}} \tag{125}$$

6.1.3 A Physical Solution

$$\psi(x,t) = \sum_{n=0}^{\infty} \frac{a^n e^{\frac{1}{4}a^2}}{\sqrt{2^n n!}} \frac{H_n(x) e^{-\imath n \omega_c t}}{\sqrt{\sqrt{\pi x_0^2} 2^n n!}} e^{-\frac{\imath}{2}\omega_c t} e^{-\frac{x^2}{2}} \tag{126}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} a e^{-\imath \omega_c t}\right)^n \frac{H_n(x)}{n!} \frac{1}{(\pi x_0^2)^{1/4}} e^{-\frac{1}{4}a^2 - \frac{1}{2}x^2 - \frac{\imath}{2}\omega_c t} \tag{127}$$

$$= \frac{1}{(\pi x_0^2)^{1/4}} e^{-\frac{1}{2}x^2 - \frac{1}{4}a^2\left(\cos^2 \omega_c t - \sin^2 \omega_c t - 2\imath \cos \omega_c t \sin \omega_c t\right)} \tag{128}$$

$$e^{a x \cos \omega_c t - \imath x a \sin \omega_c t - \frac{\imath}{2}\omega_c t} e^{-\frac{1}{4}a^2} \tag{129}$$

$$\psi(x,t) = \frac{1}{(\pi x_0^2)^{1/4}} e^{-\frac{1}{2}(x - a\cos\omega_c t)^2} e^{-\imath \phi} \tag{130}$$

where

$$\phi = \frac{1}{2}\omega_c t - \frac{a^2}{4}\sin 2\omega_c t + a x \sin \omega_c t \tag{131}$$

6.2 High-d Harmonic Oscillator

6.2.1 2-d

The Hamiltonian is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi_E(x,y) + \frac{1}{2}m\omega_0^2\left(x^2 + y^2\right)\psi_E(x,y) = E\psi_E(x,y) \tag{132}$$

This is sepratible

$$H = H_y + H_x \Rightarrow \psi_E(x,y) = \psi_{E_1}(x)\,\psi_{E_2}(y) \tag{133}$$

we now have two equations

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_{E_1}(x) + \frac{1}{2}m\omega_0^2 x^2 \psi_{E_1}(x) &= E_1 \psi_{E_1} \\ -\frac{\hbar^2}{2m}\frac{d^2}{dy^2}\psi_{E_2}(x) + \frac{1}{2}m\omega_0^2 y^2 \psi_{E_2}(y) &= E_2 \psi_{E_2} \end{aligned} \tag{134}$$

The solution is

$$\psi_{\vec{n}} = \frac{1}{\pi^{1/2}x_0}\frac{H_{n_1}\left(\frac{x}{x_0}\right)H_{n_2}\left(\frac{y}{x_0}\right)}{\sqrt{2^{n_1}n_1!}\sqrt{2^{n_2}n_2!}}e^{-\frac{x^2+y^2}{2x_0^2}} \tag{135}$$

$$E_{\vec{n}} = \hbar\omega_0\left(n_1+n_2+1\right) \tag{136}$$

note that there are degenerate energy values. For the n-th, energy levels there are $n+1$ ways to add up n_1 and n_2 to get n.

$$\begin{aligned} n &= 0+n \\ n &= 1+(n-1) \\ &\vdots \\ n &= n+0 \end{aligned}$$

$$g_{2d}\left(n\right) = n+1. \tag{137}$$

6.2.2 3-d

In this case

$$H = H_x+H_y+H_z \Rightarrow \psi_E\left(x,y,z\right) = \psi_{E_1}\left(x\right)\psi_{E_2}\left(y\right)\psi_{E_3}\left(z\right) \tag{138}$$

similar to the 2-d case. Our solution to the 3-d case is

$$\psi_{\vec{n}}\left(\vec{r}\right) = N_{n_1}N_{n_2}N_{n_3}H_{n_1}\left(\frac{x}{x_0}\right)H_{n_2}\left(\frac{y}{x_0}\right)H_{n_3}\left(\frac{z}{x_0}\right)e^{-\frac{x^2+y^2+z^2}{2x_0^2}} \tag{139}$$

$$E_{\vec{n}} = \hbar\omega_0\left(n_1+n_2+n_3+\frac{3}{2}\right) \tag{140}$$

$$\begin{aligned} g_{3d}\left(n\right) &= \sum_{n_2=0}^{n}\sum_{n_3=0}^{n-n_2} \qquad (141) \\ &= \sum_{n_2=0}^{n}\left(n-n_2+1\right) \\ &= n\left(n+1\right)+n+1-\underbrace{\sum_{n_2=0}^{n}n_2}_{\frac{1}{2}n(n+1)} \end{aligned}$$

$$g_{3d}\left(n\right) = \frac{1}{2}\left(n+1\right)\left(n+2\right) \tag{142}$$

6.3 particle in $\vec{B}$-field

6.3.1 from H to Schroed.

For a free particle in a magnetic field.

$$H = \frac{(p - qA)^2}{2m} \tag{143}$$

note that if we define $\vec{p}' \equiv \vec{p} - \frac{e\vec{A}}{c}$ this looks like a free particle.

Given that $B = B_0\hat{z}$ we have some choice in our quantity for A. I will choose A so that it points in the $\hat{y}$ direction.

$$\vec{A} = B_0 x\hat{y} \tag{144}$$

we solve the eigenvalue problem.

$$\frac{\hbar^2}{2m}\left[-\frac{\partial^2}{\partial x^2} + \left(-\imath\frac{\partial}{\partial y} - \frac{qB}{\hbar}x\right)^2 - \frac{\partial^2}{\partial z^2}\right]\psi_E = E\psi_E \tag{145}$$

this is separable

$$\psi_E(x, y, z) = \psi_{E_x}(x)\,\psi_{E_y}(y)\,\psi_{E_z}(z) \tag{146}$$

Note there is no explicit z or y dependence. So the y and z parts are just plane wave solutions.

$$-\frac{\hbar^2}{2m}\psi''_{E_x}(x) + \left(\frac{x}{\ell^2} - k_y\right)^2 \psi_{E_x}(x) = E_x\psi_{E_x}(x) \tag{147}$$

The total energy of this solution

$$E = E_x + \frac{\hbar^2}{2m}\left(k_z^2\right) \tag{148}$$

define

$$\begin{aligned} x_k &\equiv \ell^2 k_y \\ \ell^2 &\equiv \frac{h}{2\pi g B} \end{aligned} \tag{149}$$

then

$$-\frac{\hbar^2}{2m}\psi''_{E_x}(x) + \frac{\hbar^2}{2n\ell^4}(x - x_k)^2\psi_{E_x}(x) = E_x\psi_{E_x}(x) \tag{150}$$

The solution to this equation

$$\psi_{n,k_y,k_z}(x, y, z) = N_n e^{\imath k_y y + \imath k_z z} H_n\left(\frac{x - x_k}{\ell}\right) e^{-\frac{1}{2}\frac{(x-x_k)^2}{\ell^2}} \tag{151}$$

$$E_{n,k_y,k_z} = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2m}k_z^2 \tag{152}$$

$$\hbar\omega_c \equiv \frac{\hbar^2}{m\ell^2} = \frac{\hbar^2 qB}{m\hbar} \tag{153}$$

$$\omega_c = \frac{qB}{m} \tag{154}$$