Quantum Mechanics - 1: HW 4 Solutions

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1 Harmonic Oscillator

1.1 Orthonormal Wavefunctions

The wave functions for the ground and second excited state respectively are.

$$\begin{aligned}\Psi_0(x) &= e^{-\frac{x^2}{2}} && (1)\\ \Psi_2(x) &= \underbrace{H_2(x)}_{4x^2-2} e^{-\frac{x^2}{2}} && (2)\end{aligned}$$

The inner-product of these two states

$$\langle 2|0\rangle = \int_{-\infty}^{\infty} dx \left(4x^2-2\right) e^{-x^2} \tag{3}$$

$$= \sqrt{\pi}\left(4\frac{1}{2}-2\right) = 0 \tag{4}$$

1.2 Eigenstates of $1/2$-harmonic $V(x)$

Consider the potential

$$\begin{cases} \frac{1}{2}m\omega_0^2x^2 & x>0 \\ \infty & x\le 0\end{cases} \tag{5}$$

We know that $\Psi(x)=0$, when $x\le 0$. We can just use the right half of the odd states, and normalize, but exclude the even states, since they don't satisfy the boundary conditions. Thus,

$$\Psi_n(x) = \sqrt{2}H_{2n+1}(x)\,e^{-\frac{x^2}{2}} \tag{6}$$

$$E_n = \hbar\omega_0\left(2n+\frac{3}{2}\right) \tag{7}$$

1.3 Shifted Harmonic Potential

Consider the potential

$$V\left(x\right)=\frac{1}{2}m\omega_0^2\left(x^2-2cx\right) \tag{8}$$

Our Hamiltonian is

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega_0^2\left(x^2-2cx\right) \tag{9}$$

completing the square we have

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega_0^2\left(x-c\right)^2-\frac{1}{2}m\omega_0^2c^2 \tag{10}$$

we can define $y\equiv x-c$. Note that p is still the conjugate variable to y, ie.

$$\left[p,y\right]=-\imath\hbar \tag{11}$$

in terms of y

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega_0^2y^2-\frac{1}{2}m\omega_0^2c^2 \tag{12}$$

which has solutions

$$\Psi_n\left(x\right)=\sqrt{\frac{1}{\pi^{1/2}2^n n!x_0}}H_n\left(\frac{x-c}{x_0}\right)e^{-\frac{(x-c)^2}{2x_0^2}} \tag{13}$$

with energy

$$E_n=\hbar\omega_0\left(n+\frac{1}{2}\right)-\frac{1}{2}m\omega_0^2c^2 \tag{14}$$

1.4 Expectation values

For this part it'll be useful to have the following relations.

$$x=\frac{x_0}{\sqrt{2}}\left(a+a^\dagger\right) \tag{15}$$

$$p=\imath\frac{\hbar}{\sqrt{2}x_0}\left(a^\dagger-a\right) \tag{16}$$

1.4.1 x

$$\left\langle n\left|x\right|n\right\rangle=\frac{x_0}{\sqrt{2}}\left\langle n\left|\left(a+a^\dagger\right)\right|n\right\rangle \tag{17}$$

$$=\frac{x_0}{\sqrt{2}}\left(\left\langle n\left|a\right|n\right\rangle+\left\langle n\left|a^\dagger\right|n\right\rangle\right)=0 \tag{18}$$

1.4.2 p

$$\langle n\,|p|\,n\rangle = \frac{\imath\hbar}{\sqrt{2}x_0}\left\langle n\left|\left(a^\dagger - a\right)\right|n\right\rangle \tag{19}$$

$$= -\frac{\imath\hbar}{\sqrt{2}x_0}\left(\langle n\,|a|\,n\rangle - \left\langle n\left|a^\dagger\right|n\right\rangle\right) = 0 \tag{20}$$

1.4.3 x^2

$$\left\langle n\left|x^2\right|n\right\rangle = \frac{x_0^2}{2}\left\langle n\left|\left(a+a^\dagger\right)^2\right|n\right\rangle \tag{21}$$

$$= \frac{x_0^2}{2}\left(\underbrace{\left\langle n\left|\left(a^2\right)\right|n\right\rangle}_{0} + \underbrace{\left\langle n\left|\left(a^\dagger\right)^2\right|n\right\rangle}_{0} + \underbrace{\left\langle n\left|\left(aa^\dagger\right)\right|n\right\rangle}_{n+1} + \underbrace{\left\langle n\left|\left(a^\dagger a\right)\right|n\right\rangle}_{n}\right) \tag{22}$$

$$\left\langle n\left|x^2\right|n\right\rangle = x_0^2\left(n+\frac{1}{2}\right) \tag{23}$$

1.4.4 p^2

$$\left\langle n\left|p^2\right|n\right\rangle = \frac{\hbar^2}{2x_0^2}\left\langle n\left|\left(a^\dagger - a\right)^2\right|n\right\rangle \tag{24}$$

$$= -\frac{\hbar^2}{2x_0^2}\left(\underbrace{\left\langle n\left|\left(a^2\right)\right|n\right\rangle}_{0} + \underbrace{\left\langle n\left|\left(a^\dagger\right)^2\right|n\right\rangle}_{0} - \underbrace{\left\langle n\left|\left(aa^\dagger\right)\right|n\right\rangle}_{n+1} - \underbrace{\left\langle n\left|\left(a^\dagger a\right)\right|n\right\rangle}_{n}\right) \tag{25}$$

$$\left\langle n\left|p^2\right|n\right\rangle = \frac{\hbar^2}{x_0^2}\left(n+\frac{1}{2}\right) = p_0^2\left(n+\frac{1}{2}\right) \tag{26}$$

1.4.5 $\Delta x\Delta p$

$$\Delta x_n\Delta p_n = \sqrt{\left\langle x^2 - \langle x\rangle^2\right\rangle\left\langle p^2 - \langle p\rangle^2\right\rangle} \tag{27}$$

$$\Delta x_n\Delta p_n = x_0p_0\left(n+\frac{1}{2}\right) = \hbar\left(n+\frac{1}{2}\right) \tag{28}$$

1.4.6 x^4

$$\begin{aligned}\left\langle n \left| x^4 \right| n \right\rangle &= \tfrac{x_0^4}{4}\left(\left\langle n \left| \left(a^\dagger + a\right)^4 \right| n \right\rangle\right) \qquad (29)\\ &= \frac{x_0^4}{4}\left(\left\langle n \left| \left(\left(a^\dagger\right)^4 + \left(a^\dagger\right)^3 a + \left(a^\dagger\right)^2 a a^\dagger + a^\dagger a \left(a^\dagger\right)^2 + \right.\right.\right.\\ &\quad a\left(a^\dagger\right)^3 + \left(a^\dagger\right)^2 a^2 + a^\dagger a a^\dagger a + a\left(a^\dagger\right)^2 a + a^\dagger a^2 a^\dagger + a a^\dagger a a^\dagger + \qquad (30)\\ &\quad a^2\left(a^\dagger\right)^2 + a^\dagger a^3 + a a^\dagger a^2 + a^2 a^\dagger a + \\ &\quad \left.\left.\left. a^3 a^\dagger + a^4\right) \right| n \right\rangle\right)\end{aligned}$$

All terms with out an equal number a a's and $a^\dagger$'s will be zero do to the orthogonality of the $|n\rangle$'s. This leaves only six non-zero terms.

$$\left\langle n \left| x^4 \right| n \right\rangle = \frac{x_0^4}{4}\left\langle n \left| \left(\left(a^\dagger\right)^2 a^2 + a^\dagger a a^\dagger a + a a^\dagger a a^\dagger + a^2\left(a^\dagger\right)^2 + a\left(a^\dagger\right)^2 a + a^\dagger a^2 a^\dagger\right) \right| n \right\rangle \tag{31}$$

$$\left\langle n \left| x^4 \right| n \right\rangle = \frac{x_0^4}{4}\left(\left\langle n \left| \left(a^\dagger\right)^2 a^2 \right| n \right\rangle + \left\langle n \left| a^2\left(a^\dagger\right)^2 \right| n \right\rangle + n^2 + 2(n+1)n + (n+1)^2\right) \tag{32}$$

$$\tag{33}$$

we know that

$$a^2 \left|n\right\rangle = \sqrt{n\left(n-1\right)}\left|n-2\right\rangle \tag{34}$$

$$\left(a^\dagger\right)^2 \left|n\right\rangle = \sqrt{\left(n+1\right)\left(n+2\right)}\left|n+2\right\rangle \tag{35}$$

These relations imply

$$\left\langle n \left| \left(a^\dagger\right)^2 a^2 \right| n \right\rangle = n\left(n-1\right) \tag{36}$$

$$\left\langle n \left| a^2\left(a^\dagger\right)^2 \right| n \right\rangle = \left(n+1\right)\left(n+2\right) \tag{37}$$

Therefore,

$$\left\langle n \left| x^4 \right| n \right\rangle = \frac{x_0^4}{4}\left(3 + 6n\left(n+1\right)\right) \tag{38}$$

1.5 $\langle x^2 \rangle$ the hard way

This is a good way to see how cool this raising lowering operator business is. In coordinate space

$$\left\langle n \left| x^2 \right| n \right\rangle = a_n^2 \int_{-\infty}^{\infty} \frac{dx}{x_0}\frac{x^2}{x_0^2} H_n\left(\frac{x}{x_0}\right) H_n\left(\frac{x}{x_0}\right) e^{-\frac{x^2}{x_0^2}} x_0^3 \tag{39}$$

This is non-trivial, but possible by considering the following relation

$$\int_{-\infty}^{\infty} dx e^{-s^2+2sx} e^{-t^2+2tx} x^2 e^{-x^2} = \sum_{n,m} \frac{s^n t^m}{n!m!} \underbrace{\int_{-\infty}^{\infty} dx x^2 H_n(x) H_m(x) e^{-x^2}}_{I_{nm}} \tag{40}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} dx e^{-x^2+2(s+t)x} x^2 e^{-s^2-t^2} &= \frac{d^2}{dz^2} \int_{-\infty}^{\infty} dx e^{-x^2+zx} \bigg|_{z=2(s+t)} e^{-s^2-t^2} && (41)\\
&= e^{-s^2-t^2} \sqrt{\pi} \, \frac{d^2}{dz^2} e^{\frac{1}{4}z^2} \bigg|_{z=2(s+t)} && (42)\\
&= \sqrt{\pi} e^{2st} \left(\frac{1}{2} + s^2 + t^2 + 2st \right) && (43)\\
&= \sqrt{\pi} \sum_{n=0}^{\infty} \left(\frac{2^{n-1} s^n t^n}{n!} + \frac{2^n s^{n+2} t^n}{n!} + \right.\\
&\qquad \left. \frac{2^n s^n t^{n+2}}{n!} + \frac{2^{n+1} s^{n+1} t^{n+1}}{n!} \right). && (44)
\end{aligned}$$

We compare this to $\sum_{n,m} \frac{s^n t^m}{n!m!} I_{nm}$

$$\begin{aligned}
I_{nn} &= \int_{-\infty}^{\infty} dx x^2 H_n(x) H_n(x) e^{-x^2} && (45)\\
&= \sqrt{\pi} \left(2^{n-1} n! + (1-\delta_{n,0}) \frac{2^n n! n!}{(n-1)!} \right) && (46)\\
&= \sqrt{\pi} 2^n n! \left(\frac{1}{2} + n \right) && (47)
\end{aligned}$$

so,

$$\left\langle n \left| x^2 \right| n \right\rangle = A_n^2 x_0^3 I_{nn} = x_0^2 \left(n + \frac{1}{2} \right) \tag{48}$$

1.6 time evolution

Suppose $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$ then,

1.6.1 $|\Psi(t)\rangle$

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{-\frac{\imath}{2}\omega_0 t} |0\rangle + e^{-\frac{3\imath}{2}\omega_0 t} |1\rangle \right) \tag{49}$$

1.6.2 $\langle x\,(t=0)\rangle$

Making use of the Schroedinger picture

$$\begin{aligned}\langle x\,(0)\rangle &\equiv \langle \Psi\,(t=0)\,|x|\,\Psi\,(t=0)\rangle \qquad (50)\\ &= \frac{x_0}{2^{3/2}}\left(\langle 0| + \langle 1|\right)\left(a^\dagger + a\right)\left(|0\rangle + |1\rangle\right) \qquad (51)\\ &= 2^{-3/2}x_0\left(\underbrace{\langle 0\,|a|\,1\rangle}_{1} + \underbrace{\left\langle 1\,\middle|a^\dagger\middle|\,0\right\rangle}_{1}\right) \qquad (52)\end{aligned}$$

$$\langle x\,(t=0)\rangle = \frac{x_0}{\sqrt{2}} \qquad (53)$$

1.6.3 $\langle p\,(t=0)\rangle$

$$\langle p\,(t=0)\rangle = \frac{\imath\hbar}{2^{3/2}x_0}\left(\langle 0| + \langle 1|\right)\left(a^\dagger - a\right)\left(|0\rangle + |1\rangle\right) = 0 \qquad (54)$$

1.6.4 $\langle x\,(t)\rangle$

$$\begin{aligned}\langle x\,(t)\rangle &= \frac{x_0}{2^{3/2}}\left(\langle 0|\,e^{\frac{\imath}{2}\omega_0 t} + \langle 1|\,e^{\frac{3\imath}{2}\omega_0 t}\right)\left(a^\dagger + a\right)\left(e^{-\frac{\imath}{2}\omega_0 t}\,|0\rangle + e^{-\frac{3\imath}{2}\omega_0 t}\,|1\rangle\right) \qquad (55)\\ &= \frac{x_0}{2^{3/2}}\left(\langle 0\,|a|\,1\rangle\,e^{-\imath\omega_0 t} + \left\langle 1\,\middle|a^\dagger\middle|\,0\right\rangle e^{\imath\omega_0 t}\right)\langle x\,(t)\rangle \qquad (56)\\ \langle x\,(t)\rangle &= \frac{x_0}{\sqrt{2}}\cos\omega_0 t \qquad (57)\end{aligned}$$

1.6.5 $\langle p\,(t)\rangle$

We do a similar calculation

$$\langle p\,(t)\rangle = \frac{\hbar}{\sqrt{2}x_0}\frac{\imath}{2}\left(e^{\imath\omega_0 t} - e^{-\imath\omega_0 t}\right) \qquad (58)$$

$$\langle p\,(t)\rangle = \frac{-\hbar}{\sqrt{2}x_0}\sin\omega_0 t \qquad (59)$$

1.6.6 Ehrenfest's theorem

$$\langle \dot{x}\rangle = \frac{1}{\imath\hbar}\langle [x,H]\rangle = \langle p\rangle \qquad (60)$$

$$\langle \dot{p}\rangle = \frac{1}{\imath\hbar}\langle [p,H]\rangle = -m\omega_0^2\langle x\rangle \qquad (61)$$

We can use these relations to write the familiar expression

$$\langle \ddot{x}\rangle = -\omega_0^2\langle x\rangle\,. \qquad (62)$$

The solution to this differential equation is

$$\langle x(t)\rangle = \underbrace{\langle x(0)\rangle}_{\frac{x_0}{\sqrt{2}}} \cos\omega_0 t \tag{63}$$

using (60)

$$\langle p(t)\rangle = m\langle \dot{x}\rangle = -\frac{m\omega_0 x_0}{\sqrt{2}}\sin\omega_0 t \tag{64}$$

2 Two Coupled Harmonic Oscillators

Consider the Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2\left(x_1^2 + x_2^2 + (x_1 - x_2)^2\right) \tag{65}$$

2.1 Energy Spectrum

we define $y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2)$ and $y_2 = \frac{1}{\sqrt{2}}(x_1 - x_2)$ in terms of y's. This is just a rotation by $\frac{\pi}{4}$ in the $x_1 - x_2$ plane.

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2\left(y_1^2 + 3y_2^2\right) \tag{66}$$

We can rewrite this in a suggestive form

$$H = \left(\frac{p_1^2}{2m} + \frac{1}{2}m\omega_0^2 y_1^2\right) + \left(\frac{p_2^2}{2m} + \frac{1}{2}m\left(\sqrt{3}\omega_0\right)^2 y_2\right) \tag{67}$$

Now we have two *decoupled* harmonic oscillators, with frequencies ω_0, and $\sqrt{3}\omega_0$. The total energy of two independent oscillators is simply the sum of the energies.

$$E_{n_1 n_2} = \hbar\omega_0\left(n_1 + \frac{1}{2}\right) + \hbar\omega_0\sqrt{3}\left(n_2 + \frac{1}{2}\right) \tag{68}$$

we can define raising and lowering operators for each oscillator such that

$$n_1 = b_1^\dagger b_1 \tag{69}$$

$$n_2 = b_2^\dagger b_2 \tag{70}$$

The eigenstates are just

$$|n_1 n_2\rangle = \frac{\left(b_1^\dagger\right)^{n_1}\left(b_2^\dagger\right)^{n_2}}{\sqrt{n_1! n_2!}}|00\rangle \tag{71}$$

2.2 $\langle x^2 \rangle$

To make our lives easier we need to rewrite x_1^2 in terms of y_1 and y_2.

$$x_1 = \frac{1}{\sqrt{2}}\left(y_1 + y_2\right) = \frac{1}{\sqrt{2}}\left[\frac{x_0}{\sqrt{2}}\left(a_1^\dagger + a_1\right) + \frac{x_0}{\sqrt{2}3^{1/4}}\left(a_2^\dagger + a_2\right)\right] \tag{72}$$

So in product space:

$$\begin{aligned} \left\langle 00 \left| x_1^2 \right| 00 \right\rangle &= \frac{x_0^2}{4}\left\langle 0 \left| \left[\left(a_1^\dagger + a_1\right) + \frac{1}{3^{1/4}}\left(a_2^\dagger + a_2\right)\right]^2 \right| 0 \right\rangle & (73) \\ &= \frac{x_0^2}{4}\left(\left\langle 0 \left| \left(a_1^\dagger + a_1\right)^2 \right| 0 \right\rangle + \frac{1}{\sqrt{3}}\left\langle 0 \left| \left(a_2^\dagger + a_2\right)^2 \right| 0 \right\rangle + \right. \\ &\left. \frac{2}{3^{1/4}}\underbrace{\left\langle 0 \left| \left(a_1^\dagger + a_1\right)\left(a_2^\dagger + a_2\right) \right| 0 \right\rangle}_{0}\right) & (74) \\ \left\langle 00 \left| x_1^2 \right| 00 \right\rangle &= \frac{x_0^2}{4}\left(1 + \frac{1}{\sqrt{3}}\right) & (75) \end{aligned}$$

3 Baker-Hausdorff Formula

Prove

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \tag{76}$$

3.1 By Taylor Expansion

$$\begin{aligned} e^A e^B &= \left(1 + A + \frac{1}{2}A^2 + \cdots\right)\left(1 + B + \frac{1}{2}B^2 + \cdots\right) & (77) \\ &= 1 + A + B + AB + \frac{1}{2}A^2 + \frac{1}{2}B^2 + \cdots & (78) \end{aligned}$$

$$\begin{aligned} e^{A+B+\frac{1}{2}[A,B]} &= 1 + (A+B) + \frac{1}{2}(A+B)^2 + \frac{1}{2}[A,B] & (79) \\ &= 1 + A + B + \frac{1}{2}A^2 + \frac{1}{2}B^2 + \frac{1}{2}AB + \frac{1}{2}BA + \frac{1}{2}(AB - BA) & (80) \\ &= 1 + A + B + \frac{1}{2}A^2 + \frac{1}{2}B^2 + AB & (81) \end{aligned}$$

Formula is true up to quadratic order. Observe that the B before A term cancels, with the negative term of the commutator. This is true to all orders.

3.2 By Differential Equation

$$\frac{d}{dt}\left(e^{At}e^{Bt}\right) = Ae^{At}e^{Bt} + e^{At}Be^{Bt} \tag{82}$$

$$= Ae^{At}e^{Bt} + Be^{At}e^{Bt} + \left[e^{At}, B\right]e^{Bt} \tag{83}$$

$$= Ae^{At}e^{Bt} + Be^{At}e^{Bt} + t\,[A, B]\,e^{At}e^{Bt} \tag{84}$$

Here we used

$$[A, f(B)] = [A, B]\, f'(B) \tag{85}$$

true for $[A, B]$ a c-number. (This can be proved by Taylor expansion making use of $[A, B^n]$.) Define

$$O(t) \equiv e^{At}e^{Bt} \tag{86}$$

We have the differential equation

$$\frac{d}{dt}O(t) = (A + B + t\,[A, B])\,O(t) \tag{87}$$

Which has the solution

$$O(t) = e^{(A+B)t+\frac{1}{2}t^2[A,B]} \tag{88}$$

setting $t = 1$, we obtain the BCH formula, by relating (85) to (87).

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \tag{89}$$

4 Coherent States

Consider the state

$$|z\rangle = e^{za^\dagger}|0\rangle \tag{90}$$

then

4.1 $a\,|z\rangle = z\,|z\rangle$

$$ae^{za^\dagger}|0\rangle = e^{za^\dagger}\underbrace{a\,|0\rangle}_{0} + \underbrace{\left[a, e^{za^\dagger}\right]}_{ze^{za^\dagger}}|0\rangle \tag{91}$$

$$a\left(e^{za^\dagger}|0\rangle\right) = z\left(e^{za^\dagger}|0\rangle\right) \tag{92}$$

$$a\,|z\rangle = z\,|z\rangle \tag{93}$$

We can also show this using Taylor Expansion

$$|z\rangle = \sum_{n=0}^{\infty}\frac{z^n}{n!}\left(a^\dagger\right)^n|0\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle \tag{94}$$

so

$$a\left|z\right\rangle = \sum_{n=0}^{\infty} \underbrace{a\left|n\right\rangle}_{\sqrt{n}|n-1\rangle} \tag{95}$$

$$a\left|z\right\rangle = z\sum_{n=1}^{\infty} \frac{z^{n-1}}{\sqrt{(n-1)!}}\left|n-1\right\rangle \tag{96}$$

$$a\left|z\right\rangle = z\left|z\right\rangle \tag{97}$$

4.2 $\langle z_1|z_2\rangle$

$$\langle z_1|z_2\rangle = \left\langle 0\left|e^{z_1^* a}e^{z_2 a^\dagger}\right|0\right\rangle \tag{98}$$

$$= e^{z_1^* a+z_2 a^\dagger+\frac{1}{2}z_1^* z_2\underbrace{\left[a,a^\dagger\right]}_{1}} \tag{99}$$

$$= e^{z_2 a^\dagger+z_1^* a+\frac{1}{2}z_1^* z_2} \tag{100}$$

$$= e^{z_2 a^\dagger}e^{z_1^* a+\frac{1}{2}z_1^* z_2-\frac{1}{2}z_1^* z_2\left[a^\dagger,a\right]} \tag{101}$$

$$= e^{z_2 a^\dagger}e^{z_1^* a+z_1^* z_2} \tag{102}$$

$$\tag{103}$$

These z's are just c-numbers, so

$$\langle z_1|z_2\rangle = \underbrace{\left\langle 0\left|e^{z_2 a^\dagger}e^{z_1^* a}\right|0\right\rangle}_{1} e^{z_1^* z_2} \tag{104}$$

$$\langle z_1|z_2\rangle = e^{z_1^* z_2} \tag{105}$$

$$\tag{106}$$

4.3 Time Evolution

$$U\left(z,z';t\right) = \left\langle z\left|e^{-\frac{\imath}{\hbar}Ht}\right|z'\right\rangle = \left\langle z|z'\left(t\right)\right\rangle \tag{107}$$

For the HO $H=\hbar\omega_0\left(a^\dagger a+\frac{1}{2}\right)$. This implies

$$U\left(z,z';t\right) = \left\langle z\left|z' e^{-\imath\omega_0 t}\right.\right\rangle \tag{108}$$

$$= e^{z^* z' e^{-\imath\omega_0 t}} \tag{109}$$

4.4 Completeness

$$\int_{-\infty}^{\infty}\frac{dxdy}{\pi}e^{-|z|^2}\left|z\right\rangle\left\langle z\right| = \int_{-\infty}^{\infty}\frac{dxdy}{\pi}e^{-r^2}\underbrace{e^{za^\dagger}e^{\bar{z}a}\left|0\right\rangle\left\langle 0\right|}_{\sum_{n,m=0}^{\infty}\frac{z^n\bar{z}^m}{\sqrt{n!m!}}|n\rangle\langle m|} \tag{110}$$

where $z = x + \imath y = |r|\, e^{\imath\phi}$.

$$\int_{-\infty}^{\infty}\frac{dxdy}{\pi}e^{-|z|^2}\left|z\right\rangle\left\langle z\right| = \int_0^{\infty} drre^{-r^2}\sum_{n,m}^{\infty}\frac{r^{n+m}}{\sqrt{n!m!}}\int_0^{2\pi}\frac{d\phi}{\pi}e^{\imath\phi(n-m)}\left|n\right\rangle\left\langle m\right| \tag{111}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{\int_0^{\infty} d\left(r^2\right)e^{-\left(r^2\right)}\left(r^2\right)^n}_{n!}\left|n\right\rangle\left\langle n\right| \tag{112}$$

$$= \sum_{n=0}^{\infty}\left|n\right\rangle\left\langle n\right| = 1 \tag{113}$$

therefore,

$$\int_{-\infty}^{\infty}\frac{dxdy}{\pi}e^{-|z|^2}\left|z\right\rangle\left\langle z\right| = 1 \tag{114}$$

4.5 Find $\Psi_z\left(x\right)$

There are two ways to calculate this.

4.5.1 $a\left|z\right\rangle = z\left|z\right\rangle$

$$\int_{x'}\left\langle x\left|a\right|x'\right\rangle\underbrace{\left\langle x'|z\right\rangle}_{\Psi_z(x')} = z\underbrace{\left\langle x|z\right\rangle}_{\Psi_z(x)} \tag{115}$$

Recall

$$a_{xx'} = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0}+\imath\frac{p}{p_0}\right)\delta\left(x-x'\right) = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0}+x_0\frac{\partial}{\partial x}\right)\delta\left(x-x'\right) \tag{116}$$

We have the differential equation

$$\frac{1}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)\Psi_z\left(x\right) = z\Psi_z\left(x\right) \tag{117}$$

This is just a first order ODE, which is separable

$$\int\frac{d\Psi_z}{\Psi_z} = -\int_0^x\left(x-\sqrt{2}z\right)dx \tag{118}$$

so the solution is

$$\Psi_z\left(x\right) = \Psi_z\left(0\right)e^{-\left(\frac{1}{2}x^2-\sqrt{2}zx\right)} \tag{119}$$

For $z = 0$, $\Psi_0\left(0\right) = \sqrt{\frac{1}{\sqrt{\pi}x}}$. In general one can find $\Psi_z\left(0\right)$ via normalization.

$$\left\langle z|z\right\rangle = e^{|z|^2} \tag{120}$$

Also

$$\langle z|z\rangle = \int dx \Psi_z^* (x) \Psi (x) \tag{121}$$

Equating these two statements

$$e^{|z|^2} = x_0 \left|\Psi_z (0)\right|^2 \int dx e^{-x^2+2\sqrt{2}zx} = x_0 \left|\Psi_z (0)\right|^2 \sqrt{\pi} e^{2z^2} \tag{122}$$

$$\Psi_z (0) = \sqrt{\frac{1}{\sqrt{\pi}x_0}} e^{-\frac{z^2}{2}} \tag{123}$$

$$\Psi_z (x) = \sqrt{\frac{1}{\sqrt{\pi}x_0}} e^{-\frac{z^2}{2}-\frac{x^2}{2x_0^2}+\frac{\sqrt{2}zx}{x_0}} \tag{124}$$

4.6 Coherent vs Fock

$$\Psi_x (x) = \langle x|z\rangle = \left\langle x \left| e^{za^\dagger} \right| 0 \right\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \underbrace{\langle x|n\rangle}_{A_n H_n(x) e^{-\frac{x^2}{2}}} \tag{125}$$

$$= \sum_{n=0}^{\infty} \frac{z^n H_n (x) e^{-\frac{x^2}{2}}}{\sqrt{(n!)^2 x_0 \sqrt{\pi} 2^n}} \tag{126}$$

$$= \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \frac{1}{n!} H_n (x) e^{-\frac{x^2}{2}} \sqrt{\frac{1}{\sqrt{\pi}x_0}} \tag{127}$$

$$\Psi_x (x) = e^{-\frac{z^2}{2}+\sqrt{2}zx} \sqrt{\frac{1}{\sqrt{\pi}x_0}} e^{-\frac{x^2}{2}} \tag{128}$$

5 Path Integrals

Compute $U(x_f, x_i; t_f)$ for different $V(x)$ using the path integral.

5.1 $V(x) = -fx$

Then

$$U (x_f, x_i; t_f) = \int D x e^{\frac{i}{\hbar}\int_0^{t_f} dt \left(\frac{1}{2}m\dot{x}^2+fx\right)} \tag{129}$$

and we can write every solution

$$x (t) = x_c (x) + y (t) \tag{130}$$

where x_c is the classical solution which obeys the equation of motion

$$m\ddot{x}_c (t) = f \tag{131}$$

Which has the solution

$$x_c\left(t\right) = \frac{1}{2}\frac{f}{m}t^2 + v_0 t + x_i \tag{132}$$

Since x_f, x_i, and t_f we can write v_0 in terms of known quantities

$$v_0 = \frac{1}{t_f}\left(x_f - x_i - \frac{1}{2}\frac{f}{m}t_f^2\right) \tag{133}$$

In previous HW assignments we showed that the cross terms from $\dot{x}^2$ are zero via the equations of motion. Therefore we can separate this integral into the sum of two terms containing y or x_c exclusively. ie.

$$S\left[x\left(t\right)\right] = S\left[x_c\left(t\right)\right] + S\left[y\left(t\right)\right] \tag{134}$$

where

$$S\left[x_c\left(t\right)\right] = \int_0^{t_f} dt\left(\frac{1}{2}m\dot{x}_c^2 + fx_c\right) \tag{135}$$

$$S\left[y\left(t\right)\right] = \int_0^{t_f} dt\frac{1}{2}m\dot{y}^2 \tag{136}$$

Here I'll simplify $S\left[x_c\right]$. making use of equation (133)

$$\begin{aligned} S\left[x_c\left(t\right)\right] &= \int_0^{t_f} dt\left[\frac{1}{2}m\left(\frac{f}{m}t + v_0\right)^2 + \frac{f^2}{2m}t^2 + fv_0 t + fx_i\right] && (137)\\ &= \int_0^{t_f} dt\left[\frac{2f^2}{2m}t^2 + 2fv_0 t + \frac{1}{2}mv_0^2 + fx_i\right] && (138)\\ &= \frac{f^2}{3m}t_f^3 + f\left(x_f - x_i - \frac{f}{2m}t_f^2\right)t_f + \left[\frac{1}{2}\frac{m}{t_f^2}\left(x_f - x_i - \frac{f}{2m}t_f^2\right)^2 + fx_i\right]t_f && (139)\\ &= \frac{f^2}{3m}t_f^3 - \frac{f^2}{2m}t_f^3 + \frac{f^2}{8m}t_f^3 + \frac{m}{2t_f}\left(x_f - x_i\right)^2 + ft_f x_i + \frac{1}{2}f\left(x_f - x_i\right)t_f && (140) \end{aligned}$$

$$S\left[x_c\left(t\right)\right] = -\frac{f^2}{24m}t_f^3 + \frac{1}{2}\frac{m\left(x_f - x_i\right)^2}{t_f} + \frac{1}{2}ft_f\left(x_f + x_i\right) \tag{141}$$

We can see that in the limit $f \to 0$ that S_c reduces to the free particle solution $S_c = \frac{1}{2}m\frac{\left(x_f - x_i\right)^2}{t_f}$. The integral over the non-classical part is familiar.

$$\int_0^0 Dy\left(t\right)e^{\frac{\imath}{\hbar}\int_0^{t_f} dt\frac{1}{2}m\dot{y}^2} = \sqrt{\frac{m}{2\pi\hbar\imath t}} \tag{142}$$

The full propagator is

$$U\left(x_f, x_i; t_f\right) = \sqrt{\frac{m}{2\pi\hbar\imath t}}e^{\frac{\imath}{\hbar}S_c} \tag{143}$$

5.2 Harmonic Oscillator

The potential

$$V(x) = \frac{1}{2}m\omega_0^2 x^2 \tag{144}$$

Like before we separate S into a classical and non-classical part.

$$S[x(x)] = S[x_c(t)] + S[y(t)] \tag{145}$$

We have done S_c before in HW 1 problem 2. We quote the result here.

$$S[x_c(t)] = \frac{m\omega_0}{2\sin\omega_0 t_f}\left[\left(x_f^2 + x_i^2\right)\cos\omega_0 t_f - 2x_i x_f\right] \tag{146}$$

now we calculate the non-classical contribution.

$$A_{\omega_0}(t_f) = \int_0^0 Dy(t)\, e^{\frac{\imath}{\hbar}\int_0^{t_f} dt\left[\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega_0^2 y^2\right]} \tag{147}$$

because we require $y_i = Y_f = 0$ we can expand y

$$y(t) = \sum_n y_n \sin\omega_n t, \text{ where } \omega_n \equiv \frac{\pi n}{t_f} \tag{148}$$

To sum over all the paths one must just integrate over every Fourier coefficient. (This comes from (1) that two different sets of Fourier coefficients lead to two different solutions, and (2) that due to B.C.'s every suitable function can be written as a Fourier series.)

$$A_{\omega_0}(t_f) = \prod_{n=1}^{\infty}\int dy_n e^{\frac{\imath}{\hbar}S[y(t)]}J \tag{149}$$

Where J is the Jacobian coming from the change in variables Finding $S[y]$ in terms of Fourier coefficients

$$\begin{aligned} S[y(t)] &= \int_0^{t_f} dt\left[\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega_0^2 y^2\right] && (150)\\ &= \sum^{n,m}\frac{1}{2}m\omega_n\omega_m y_n y_m \underbrace{\int_0^{t_f} dt\cos\omega_n t\cos\omega_m t}_{\frac{1}{2}t_f\delta_{nm}} && (151)\\ &\quad -\sum\frac{1}{2}m\omega_0^2 y_n y_m \underbrace{\int_0^{t_f} dt\sin\omega_n t\sin\omega_m t}_{\frac{1}{2}t_f\delta_{nm}} && (152)\\ S[y(t)] &= \sum_{n=1}^{\infty}\frac{t_f}{4}m\left(\omega_n^2 - \omega_0^2\right)y_n^2 && (153)\end{aligned}$$

Plugging this into (149)

$$
\begin{aligned}
A_{\omega_0}\left(t_f\right) &= J \lim_{N\to\infty} \prod_{n=1}^{N} \left(\int dy_n e^{\frac{\imath}{\hbar}\frac{t_f}{4}m\left(\omega_n^2-\omega_0^2\right)y_n^2}\right) && (154)\\
&= J \lim_{N\to\infty} \prod_{n=1}^{N} \sqrt{\frac{4\pi\hbar\ \imath}{t_f m\left(\omega_n^2-\omega_0^2\right)}} && (155)\\
&= \underbrace{J \prod_{n=1}^{\infty} \sqrt{4\pi\hbar\imath/t_f m\omega_n^2}}_{C \text{ 'some constant'}} \underbrace{\prod_{n=1}^{\infty} \frac{1}{\sqrt{1-\frac{\omega_0^2}{\omega_n^2}}}}_{=\sqrt{\omega_0 t_f}\sin\omega_0 t_f} && (156)\\
A_{\omega_0}\left(t_f\right) &= C\sqrt{\omega_0 t_f}\sin\omega_0 t_f && (157)
\end{aligned}
$$

We find C by appealing to the $\omega_0 \to 0$ limit, which should reduce to the free particle case.

$$
\begin{aligned}
U\left(x_f, x_i; t_f\right) &\to C e^{\frac{\imath}{\hbar}\frac{m\left(x_f-x_i\right)^2}{2t_f}} && (158)\\
\Rightarrow C &= \sqrt{\frac{m}{2\pi\imath\hbar t_f}} && (159)
\end{aligned}
$$

Finally

$$
U_{\omega_0}\left(x_f, x_i; t_f\right) = \sqrt{\frac{m\omega_0}{2\pi\imath\hbar\sin\omega_0 t_f}}\, e^{\frac{\imath m\omega_0}{2\hbar\sin\omega_0 t_f}\left[\left(x_f^2-x_i^2\right)\cos\omega_0 t_f-2x_i x_f\right]} \tag{160}
$$