

Quantum Mechanics - 1: HW 6 Solutions

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1 Problem 1

Show that $H(\vec{E}_1 \cdot \vec{E}_2)$ commutes with \vec{L} , where $\vec{E}_{1,2}$ are vector operators.

The definition of a vector operator.

$$\left[E_i^{(1 \text{ or } 2)}, L_j \right] = i\hbar \epsilon_{ijk} E_k \quad (1)$$

(transforms like a vector under rotations)

$$\left[E_i^{(1)} E_i^{(2)}, L_j \right] = E_i^{(1)} \left[E_i^{(2)}, L_j \right] + \left[E_i^{(1)}, L_j \right] E_i^{(2)} \quad (2)$$

$$= i\hbar \epsilon_{ijk} \left(E_i^{(1)} E_k^{(2)} + E_k^{(1)} E_i^{(2)} \right) \quad (3)$$

$$= 0 \quad (4)$$

Because $\vec{E}_1 \cdot \vec{E}_2$ commute with \vec{L} all functions of $\vec{E}_1 \cdot \vec{E}_2$ commute with \vec{L}

$$\left[H(\vec{E}_1 \cdot \vec{E}_2), \vec{L} \right] = 0 \quad (5)$$

2 Problem 2

Show that $L_z = -i\hbar(x\partial_y - y\partial_x) = -i\hbar\partial_\phi$ relating coordinates

$$x = r \sin \theta \cos \phi \quad (6)$$

$$y = r \sin \theta \sin \phi \quad (7)$$

$$z = r \cos \theta \quad (8)$$

and derivatives

$$\partial_y = \frac{\partial r}{\partial y} \partial_r + \frac{\partial \theta}{\partial y} \partial_\theta + \frac{\partial \phi}{\partial y} \partial_\phi \quad (9)$$

$$\partial_x = \frac{\partial r}{\partial x} \partial_r + \frac{\partial \theta}{\partial x} \partial_\theta + \frac{\partial \phi}{\partial x} \partial_\phi \quad (10)$$

$$r = \sqrt{x^2 + y^2 + z^2} \quad (11)$$

$$\frac{z}{r} = \cos \theta \quad (12)$$

calculating derivatives

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin \theta \cos \phi \quad (13)$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta \sin \phi \quad (14)$$

θ derivatives can be taken easily by making use of the chain rule and equation (12)

$$\frac{\partial}{\partial x} \frac{z}{r} = -\frac{z}{r^2} \frac{\partial r}{\partial x} = -\sin \theta \frac{\partial \theta}{\partial x} = -\sin \theta \frac{\partial \theta}{\partial x} \quad (15)$$

solving for $\frac{\partial \theta}{\partial x}$.

$$\frac{\partial \theta}{\partial x} = \frac{1}{r} \cos \theta \cos \phi \quad (16)$$

the calculation for $\frac{\partial \theta}{\partial y}$ is almost identical

$$\frac{\partial \theta}{\partial y} = \frac{1}{r} \cos \theta \sin \phi \quad (17)$$

finding ϕ derivatives

$$\frac{y}{x} = \tan \phi \quad (18)$$

taking derivatives of both sides

$$-\frac{\tan \phi}{x} = \sec^2 \phi \frac{\partial \phi}{\partial x} \quad (19)$$

solving for $\frac{\partial \phi}{\partial x}$.

$$\frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r \sin \theta} \quad (20)$$

a similar argument can be used for y

$$\frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r \sin \theta} \quad (21)$$

Putting all of it together

$$\frac{L_z}{-i\hbar} = x\partial_y - y\partial_x = r \sin \theta \cos \phi (\partial_y r \partial_r + \partial_y \theta \partial_\theta + \partial_y \phi \partial_\phi) \quad (22)$$

$$-r \sin \theta \sin \phi (\partial_x r \partial_r + \partial_x \theta \partial_\theta + \partial_x \phi \partial_\phi)$$

$$L_z = -i\hbar \partial_\phi \quad (23)$$

3 Problem 3

Suppose we have three identical bosonic particles constrained to move on a circle in an equilateral configuration and want to find the wavefunction and the energy spectrum. The Hamiltonian of this system is

$$H = \frac{L_z^2}{2I} \quad (24)$$

where

$$I = 3mr^2 \quad (25)$$

The eigenstates of this Hamiltonian are the eigenstates of the L_z operator.

$$L_z |\ell_z\rangle = \hbar \ell_z |\ell_z\rangle \quad (26)$$

$$\phi_{\ell_z}(\phi) = e^{i\ell_z\phi} \quad (27)$$

The position of each of the particles are

$$\phi_1 = \phi \quad (28)$$

$$\phi_2 = \phi + \frac{2\pi}{3} \quad (29)$$

$$\phi_3 = \phi + \frac{4\pi}{3} \quad (30)$$

Because these are bosons, we must require that the full wavefunction be symmetric under permutations. For the rigid structure above this amounts to $\phi_m(\phi \rightarrow \phi + \frac{2\pi}{3}) = \phi_m(\phi)$. Therefore, $\ell_z = 3n$, where $n \in \mathbb{Z}$.

$$\phi_n(\phi) = e^{3in\phi} \quad (31)$$

$$E_n = \frac{9\hbar^2 n^2}{2I} \quad (32)$$

4 Problem 4

Express Y_ℓ^m 's in Cartesian coords

4.1 Y_0^0

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \quad (33)$$

4.2 $Y_1^{\pm 1}$

$$Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta \underbrace{e^{\pm i\phi}}_{\sin \theta \cos \phi \pm i \sin \theta \sin \phi} \quad (34)$$

$$= \mp \sqrt{\frac{3}{8\pi}} \left(\frac{x \pm iy}{\sqrt{x^2 + y^2 + z^2}} \right) \quad (35)$$

4.3 Y_1^0

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta = \sqrt{\frac{3\pi}{4}} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \quad (36)$$

4.4 $Y_2^{\pm 2}$

$$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta (\cos 2\phi \pm i \sin 2\phi) \quad (37)$$

$$= \sqrt{\frac{15}{32\pi}} \sin^2 \theta (\cos^2 \phi - \sin^2 \phi \pm 2i \sin \phi \cos \phi) \quad (38)$$

$$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \left(\frac{x^2 - y^2 \pm 2ixy}{x^2 + y^2 + z^2} \right) \quad (39)$$

4.5 $Y_2^{\pm 1}$

$$Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta (\cos \phi \pm i \sin \phi) \quad (40)$$

$$Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \left(\frac{xz \pm iyz}{x^2 + y^2 + z^2} \right) \quad (41)$$

4.6 Y_2^0

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} \left(3 \frac{z^2}{x^2 + y^2 + z^2} - 1 \right) \quad (42)$$

5 Problem 5

5.1 a:

$$\psi(\rho, \phi) = \underbrace{Ae^{-\frac{\rho^2}{\rho_0^2}}}_{u(\rho)} \cos^2 \phi \quad (43)$$

project out C_{ℓ_z} via Y_ℓ^m 's.

$$N^2 |C_{\ell_z}|^2 \Rightarrow N^2 (1 + 4 + 1) = 1 \Rightarrow N^2 = \frac{1}{6} \quad (44)$$

$$P(\ell_z) = \begin{cases} \frac{1}{6} & \ell_z = \pm 2 \\ \frac{2}{3} & \ell_z = 0 \\ 0 & \text{else} \end{cases} \quad (45)$$

5.2 b:

$$\psi(x, y, z) = A(xy + yz + xz) e^{-\frac{r}{r_0}} \quad (46)$$

use the result from problem 4:

$$xy = \frac{-i}{4} \sqrt{\frac{32\pi}{15}} (Y_2^2 - Y_2^{-2}) r^2 \quad (47)$$

$$yz = \frac{i}{2} \sqrt{\frac{8\pi}{15}} (Y_2^1 + Y_2^{-1}) r^2 \quad (48)$$

$$xz = -\frac{1}{2} \sqrt{\frac{8\pi}{15}} (Y_2^1 - Y_2^{-1}) r^2 \quad (49)$$

$$(50)$$

therefore

$$\psi(x, y, z) = \frac{-A}{2} e^{-\frac{r}{r_0}} \sqrt{\frac{8\pi}{15}} [iY_2^2 + (1-i)Y_2^1 - (1+i)Y_2^{-1} - iY_2^{-2}] r^2 \quad (51)$$

via orthogonality of the Y_ℓ^m 's

$$C_{2,2} = N i \quad \Rightarrow P_{2,2} = N^2 = \frac{1}{6} \quad (52)$$

$$C_{2,1} = N(1-i) \quad \Rightarrow P_{2,1} = 2N^2 = \frac{1}{3} \quad (53)$$

$$P_{2,0} = 0 \quad (54)$$

$$C_{2,-1} = N(1+i) \quad \Rightarrow P_{2,-1} = 2N^2 = \frac{1}{3} \quad (55)$$

$$C_{2,-2} = -iN \quad \Rightarrow P_{2,-2} = N^2 = \frac{1}{6} \quad (56)$$

all other coefficients are 0.

6 Problem 6

Rotation of $\ell = 1$ (vector) eigenstates. $\theta_x \hat{x}$ rotation:

$$x \rightarrow x \quad (57)$$

$$y \rightarrow y \cos \theta_x - z \sin \theta_x \quad (58)$$

$$z \rightarrow z \cos \theta_x + y \sin \theta_x \quad (59)$$

$$(60)$$

$$\psi(x, y, z) \rightarrow \psi(x, y \cos \theta_x - z \sin \theta_x, z \cos \theta_x + y \sin \theta_x) \quad (61)$$

check on $\psi = Aze^{-\frac{r^2}{a^2}} \rightarrow A(z \cos \theta_x - y \sin \theta_x) e^{-\frac{r^2}{a^2}}$ by working in Y_ℓ^m representation.

$$\psi = Ae^{-\frac{r^2}{a^2}} z = A \sqrt{\frac{4\pi}{3}} r e^{-\frac{r^2}{a^2}} Y_1^0 \quad (62)$$

first we write this as a vector

$$\psi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (63)$$

then we rotate using the rotation matrix

$$e^{-i\theta_x L_x/\hbar} = (\cos\theta_x - 1) \left(\frac{L_x}{\hbar}\right)^2 - i \sin\theta_x \left(\frac{L_x}{\hbar}\right) + I \quad (64)$$

because $L_x^3 = L_x$. In matrix form

$$e^{-i\theta_x L_x/\hbar} = (\cos\theta_x - 1) \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}^2 - i \sin\theta_x \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad (65)$$

As a check you should convince yourself that $x \rightarrow x$ here.

$$e^{i\theta_x L_x/\hbar} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - i \sin\theta_x \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} + (\cos\theta_x - 1) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (66)$$

$$= \cos\theta_x Y_1^0 - i \sin\theta_x \frac{1}{\sqrt{2}} (Y_1^1 + Y_1^{-1}) \quad (67)$$

$$= \frac{1}{r} \sqrt{\frac{3}{4\pi}} (\cos\theta_x z - \sin\theta_x y) \quad (68)$$

therefore

$$\psi \rightarrow A \sqrt{\frac{4\pi}{3}} r e^{-\frac{r^2}{a^2}} \frac{1}{r} \sqrt{\frac{3}{4\pi}} (\cos\theta_x z - \sin\theta_x y) = A (\cos\theta_x z - \sin\theta_x y) e^{-\frac{r^2}{a^2}} \quad (69)$$

7 Problem 7

Harmonic Oscillator

7.1 a: 2D in polar

$$H = \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2\rho^2 \quad (70)$$

7.1.1 i: $[H, L_z]$

One can just use answer to problem 1. WE do it here explicitly.

$$[p_x^2 + p_y^2, xp_y - yp_x] = [p_x^2, x] p_y + [p_y^2, x] p_y - [p_x^2, y] p_x - [p_y^2, y] p_x \quad (71)$$

$$= 2p_x [p_x, x] p_y - 2p_y [p_y, y] p_x \quad (72)$$

$$= -2i\hbar (p_x p_y - p_y p_x) = 0 \quad (73)$$

therefore

$$[H, L_z] = 0 \quad (74)$$

So eigenfunctions of H are eigenfunctions of L_z . Keep this in mind as we look at the HO Hamiltonian.

$$H\psi = E\psi \quad (75)$$

$$\left[-\frac{\hbar^2}{2\mu} \nabla^2 + V(\rho) \right] \psi = E\psi \quad (76)$$

$$\left[-\frac{\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) + V(\rho) \right] R(\rho) \Phi(\phi) = ER(\rho) \Phi(\phi) \quad (77)$$

We extract the angular dependence by taking $\Phi = \Phi_m$, where $L_z \Phi_m = \hbar m \Phi_m$, with m integers. H in terms of L_z . Note that $\Phi_m = e^{im\phi}$, forms a complete set.

$$\left[-\frac{\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) + \frac{L_z^2}{2\mu\rho^2} + V(\rho) \right] R(\rho) \Phi_m(\phi) = ER(\rho) \Phi_m(\phi) \quad (78)$$

$$\left[-\frac{\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) + \left(\frac{1}{2} \mu \omega^2 \rho^2 + \frac{\hbar^2 m^2}{2\mu\rho^2} \right) \right] R(\rho) = ER(\rho) \quad (79)$$

7.1.2 ii: $\rho \rightarrow 0$ limit

In the $\rho \rightarrow 0$ the effective potential is dominated by a centrifugal barrier, due to angular momentum conservation. ($V_{eff} \rightarrow \frac{m^2}{\rho^2}$)

$$-\frac{\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} \right) R_m(\rho) = ER_{E,m}(\rho) \approx 0 \quad (80)$$

suppose

$$R_m(\rho) = \rho^\alpha \quad (81)$$

we get a relation for α using eqn (80).

$$\alpha(\alpha - 1) + \alpha - m^2 = 0 \Rightarrow \alpha = \pm m \quad (82)$$

We expect our wavefunction to be well behaved at $\rho = 0$, so we'll take $\alpha = |m|$.

$$\lim_{\rho \rightarrow 0} R_{E,m} = C \rho^{|m|} \quad (83)$$

7.1.3 iii: $\rho \rightarrow \infty$ limit

In this limit we drop the $\frac{m^2}{\rho^2}$, E , and $\frac{1}{\rho} \frac{\partial}{\partial \rho}$ terms.

$$\left(-\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial \rho^2} + \frac{1}{2} \mu \omega^2 \rho^2 \right) R_m(\rho) = 0 \quad (84)$$

solution to this is

$$e^{-\frac{\rho^2}{2\rho_0^2}}, \text{ where } \rho_0 = \sqrt{\frac{\hbar}{\mu\omega}} \quad (85)$$

7.1.4 iv: dimensionless variables

define

$$x \equiv \frac{\rho}{\rho_0} \quad (86)$$

$$\epsilon \equiv \frac{E}{\hbar\omega} \quad (87)$$

$$\left[-\underbrace{\frac{\hbar^2}{2\mu}}_{\frac{1}{2}\hbar\omega} \frac{1}{\rho_0^2} \left(\partial_x^2 + \frac{1}{x} \partial_x \right) + \underbrace{\frac{1}{2} \mu \omega^2 \rho_0^2}_{\frac{1}{2}\hbar\omega} x^2 + \underbrace{\frac{\hbar^2 m^2}{2\mu \rho_0^2}}_{\frac{1}{2}\hbar\omega m^2} \frac{1}{x^2} \right] R_{E,m}(\rho_0 x) = E R_{E,m}(\rho_0 x) \quad (88)$$

$$\left(\partial_x^2 + \frac{1}{x} \partial_x \right) R_m - \left(x^2 + \frac{m^2}{x^2} \right) R_m = -2\epsilon R_m \quad (89)$$

suppose

$$R_m = x^{|m|} f(x) \quad (90)$$

The differential equation becomes

$$\begin{aligned} & |m| (|m| - 1) x^{|m|-2} f(x) + 2|m| \partial_x f(x) x^{|m|-1} + x^{|m|} \partial_x^2 f(x) + |m| x^{|m|-2} f(x) \\ & + x^{|m|-1} \partial_x f(x) - \left(x^{|m|+2} + m^2 x^{|m|-2} \right) f(x) + 2\epsilon x^{|m|} f(x) = 0 \end{aligned} \quad (91)$$

$$\partial_x^2 f(x) + \frac{2|m|+1}{x} \partial_x f(x) + (2\epsilon - x^2) f(x) = 0 \quad (92)$$

7.1.5 v: $f(x) = u(x)e^{-\frac{x^2}{2}}$

This is required to meet the $x \rightarrow \infty$ limit.

$$\partial_x f = e^{-\frac{x^2}{2}} u' - x e^{-\frac{x^2}{2}} u \quad (93)$$

$$\partial_x^2 f = e^{-\frac{x^2}{2}} u'' - 2x e^{-\frac{x^2}{2}} u' + x^2 e^{-\frac{x^2}{2}} u - e^{-\frac{x^2}{2}} u \quad (94)$$

equation for u

$$u'' - 2xu' - u + x^2 u + \frac{2|m|+1}{x} (u' - xu) + (2\epsilon - x^2) u = 0 \quad (95)$$

$$u'' + \left[\frac{2|m|+1}{x} - 2x \right] u' + (2\epsilon - 2|m| - 2) u = 0 \quad (96)$$

7.1.6 vi: A series solution

The large x limit is already taken care of by the exponential. Suppose our solution has the form

$$u(x) = \sum_{n=0}^{\infty} c_n x^n \quad (97)$$

From the differential equation we get a recursion relation for our coefficients c_n .

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)x^{n-2}c_n + (2|m|+1)\sum_{n=1}^{\infty} nc_nx^{n-2} - 2\sum_{n=1}^{\infty} nc_nx^n + \\ (2\epsilon - 2|m| - 2)\sum_{n=0}^{\infty} c_nx^n = 0 \end{aligned} \quad (98)$$

$$\begin{aligned} \sum_{n=0}^{\infty} (n+2)(n+1)x^n c_{n+2} + (2|m|+1)\sum_{n=1}^{\infty} (n+2)c_{n+2}x^n - \\ 2\sum_{n=1}^{\infty} nc_nx^n + (2\epsilon - 2|m| - 2)\sum_{n=0}^{\infty} c_nx^n = 0 \end{aligned} \quad (99)$$

$$\begin{aligned} \sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + 2(\epsilon - |m| - 1)c_n - 2nc_n + \\ (n+2)(2|m|+1)c_{n+2}]x^n = 0 \end{aligned} \quad (100)$$

To get above-deduced $x^{|m|}$ small x asymptotics for $R_m(x)$, we must take $c_1 = 0$. This is consistent with the small r limit we found in part ii.

$$c_{n+2} = 2 \frac{(-\epsilon + |m| + 1 + n)}{(n+2)(n+2|m|+2)} c_n \quad (101)$$

7.1.7 vii: terminating the series

Note that for large n , $\frac{c_{n+2}}{c_n} \rightarrow \frac{2}{n}$. Compare this to $e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \dots$. For this series $\frac{c_{2k+2}}{c_{2k}} = \frac{k!}{(k+1)!} = \frac{1}{k+1}$. For large k this also converges to $\frac{n}{2}$. Therefore if our series does not terminate at some n then our solution will be $O\left(e^{\frac{x^2}{2}}\right)$. If we assume that $\epsilon_p = |m| + 1 + p$, where $p \in \mathbb{Z}$, then the series terminates. Because $C_1 = 0$, $p = 2k$, where $k \in \mathbb{Z}$. This quantizes our energy

$$\hbar\omega(n+1), n \equiv |m| + 2k \quad (102)$$

7.1.8 viii: degeneracy

For a given energy E_n

$$|m| = n - 2k \quad (103)$$

m is even/odd when n is even/odd

$$|m| = \begin{cases} n, n-2, n-4, \dots, 0 & n \in \text{even} \\ n, n-2, n-4, \dots, 1 & n \in \text{odd} \end{cases} \quad (104)$$

This means the degeneracy of E_n is $n+1$. This agrees with our treatment in Cartesian coordinates.

$$e = \hbar\omega \left(\underbrace{n_x + n_y}_n + 1 \right) \quad (105)$$

7.1.9 ix: Eigenfunctions

$n = 0 \Rightarrow m = 0, D_0 = 1$

$$\phi_{0,0}(\rho, \phi) = N_{0,0} e^{-\frac{\rho^2}{2\rho_0}} \quad (106)$$

$n = 1 \Rightarrow m \in \{1, 0, -1\}, D_1 = 2$

$$u_1 = c_0 + c_2 x^2, c_2 = 0 \quad (107)$$

$$\phi_{1,\pm 1}(\rho, \phi) = N_{1,\pm 1} e^{-\frac{\rho^2}{2\rho_0}} \left(\frac{\rho}{\rho_0} \right) e^{\pm i\phi} \quad (108)$$

$n = 2 \Rightarrow m \in \{2, 0, -2\}, D_2 = 3$

$$u_{2,2}(x) = c_0^{(2)} + c_2^{(2)} x^2 \quad (109)$$

$$c_2^{(2)} = 2(-3 + 2 + 1 + 0) c_0^{(2)} = 0 \quad (110)$$

$$u_{2,0}(x) = c_0^{(0)} + c_2^{(0)} x^2 \quad (111)$$

$$c_2^{(0)} = 2 \frac{-3 + 1 + 0}{2 * 2} c_0^{(0)} = -c_0^{(0)} \quad (112)$$

$$\phi_{2,\pm 2}(\rho, \phi) = N_{2,\pm 2} e^{-\frac{\rho^2}{2\rho_0}} \left(\frac{\rho}{\rho_0} \right)^2 e^{\pm 2i\phi} \quad (113)$$

$$\phi_{2,0}(\rho, \phi) = N_{2,0} e^{-\frac{\rho^2}{2\rho_0}} \left(1 - \frac{\rho^2}{\rho_0^2} \right) \quad (114)$$

7.1.10 x: relating Cartesian and polar coordinate solutions

$n = 0$ state is non-degenerate so it must match the Cartesian form exactly

$$\phi_{0,0} \propto e^{-\frac{\rho}{2\rho_0}} = e^{-\frac{x^2}{2\rho_0}} e^{-\frac{y^2}{2\rho_0}} \quad (115)$$

$n = 1$ states are two-fold degenerate so the a Cartesian state must just be some linear combination of the old states

$$\phi_{1,\pm 1} \propto \rho e^{-\frac{\rho^2}{2\rho_0^2}} e^{\pm i\phi} \quad (116)$$

$$\propto (x \pm iy) e^{-\frac{\rho^2}{2\rho_0^2}} \quad (117)$$

$$\propto \left(H_1(x) e^{-x^2/2\rho_0} \right) \left(H_0(y) e^{-\frac{y^2}{2\rho_0}} \right) \pm i \left(H_0(x) e^{-\frac{x^2}{2\rho_0}} \right) \left(H_1(y) e^{-\frac{y^2}{2\rho_0}} \right) \quad (118)$$

$$\phi_{2,\pm 2} \propto \left(\underbrace{\rho^2 \cos 2\phi}_{x^2 - y^2} \pm i \underbrace{\rho^2 \sin 2\phi}_{2xy} \right) e^{-\frac{\rho^2}{2\rho_0^2}} \quad (119)$$

$$\begin{aligned} & \propto H_2(x) e^{-\frac{x^2}{2\rho_0}} H_0(y) e^{-\frac{y^2}{2\rho_0}} - H_0(x) e^{-\frac{x^2}{2\rho_0}} H_2(y) e^{-\frac{y^2}{2\rho_0}} \\ & \pm i 2H_1(x) e^{-\frac{x^2}{2\rho_0}} H_1(y) e^{-\frac{y^2}{2\rho_0}} \end{aligned} \quad (120)$$

$$\phi_{2,0} \propto \left(1 - \frac{x^2 + y^2}{\rho_0^2} \right) e^{-\frac{\rho^2}{2\rho_0^2}} \quad (121)$$

$$\propto H_2(x) e^{-\frac{x^2}{2\rho_0}} H_0(y) e^{-\frac{y^2}{2\rho_0}} + H_0(x) e^{-\frac{x^2}{2\rho_0}} H_2(y) e^{-\frac{y^2}{2\rho_0}} \quad (122)$$

Parity from before was $(-1)^n$. Since odd/even m corresponds to odd/even n , parity in the polar representation is $(-1)^m$.

7.2 b: Rotating system

If we rotate the system with frequency Ω , then

$$H \rightarrow H_\Omega = \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2\rho^2 - \Omega L_z \quad (123)$$

$$E_{n,m} = \hbar\omega(n+1) - \hbar\Omega m \quad (124)$$

For small $\Omega < \omega$, we get a small splitting of previously degenerate energy levels. For $\Omega \geq \omega$, the system becomes unstable as a result of centrifugal kinetic energy dominating the potential.

$$V_{eff} = \frac{1}{2}m\omega^2 r^2 - \frac{1}{2}m\Omega^2 r^2 \quad (125)$$

7.3 c: 3D Oscillator

$$H = \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 r^2 \quad (126)$$

The eigenfunctions have the form

$$\phi_{E,\ell,m} = \frac{U_{E,\ell}}{r} Y_\ell^m \quad (127)$$

$$E_n = \hbar\omega \left(n + \frac{3}{2} \right) \quad (128)$$

These states are degenerate. Each n, ℓ has $2\ell + 1$ m -states. For a given n the allowed values of ℓ are given by

$$\ell = n - 2k, \text{ where } k \in \mathbb{Z} \quad (129)$$

7.3.1 i: degeneracy

$$D_n = \sum_{k=0}^{n/2} \sum_{\ell=0}^n \underbrace{\sum_{m=-\ell}^{\ell} \delta_{\ell, n-2k}}_{2\ell+1} \quad (130)$$

$$= \sum_{k=0}^{n/2} [2(n - 2k) + 2] \quad (131)$$

$$= \sum_{k=0}^{n/2} (2n + 1) - 4 \sum_{k=0}^{n/2} k \quad (132)$$

$$= (2n + 1) \left(\frac{n}{2} + 1 \right) - n \left(\frac{n}{2} + 1 \right) = (n + 1) \left(\frac{n}{2} + 1 \right) \quad (133)$$

$$D_n = \frac{(n + 1)(n + 2)}{2} \quad (134)$$

7.3.2 ii: eigenfunctions

$$\psi_{0,0,0}(r, \theta, \phi) = N_{0,0} e^{-\frac{r^2}{2r_0^2}} = e^{-\frac{x^2}{2r_0^2}} e^{-\frac{y^2}{2r_0^2}} e^{-\frac{z^2}{2r_0^2}} \quad (135)$$

$$\psi_{1,1,0}(r, \theta, \phi) = N_{1,1,0} e^{-\frac{r^2}{2r_0^2}} \underbrace{Y_{1,0}(\theta, \phi)}_z \quad (136)$$

$$= z e^{-\frac{r^2}{2r_0^2}} \quad (137)$$

$$\psi_{1,1,0}(r, \theta, \phi) = e^{-\frac{x^2}{2r_0^2}} e^{-\frac{y^2}{2r_0^2}} H_1(z) \quad (138)$$

$$\psi_{1,1,\pm 1}(r, \theta, \phi) = N_{1,1,\pm 1} e^{-\frac{r^2}{2r_0^2}} r Y_{1,\pm 1}(\theta, \phi) \quad (139)$$

$$= e^{-\frac{r^2}{2r_0^2}} r \sin \theta (\cos \theta \pm i \sin \phi) \quad (140)$$

$$\psi_{1,1,\pm 1}(r, \theta, \phi) = x e^{-\frac{x^2}{2r_0^2}} e^{-\frac{y^2}{2r_0^2}} e^{-\frac{z^2}{2r_0^2}} \pm i e^{-\frac{x^2}{2r_0^2}} y e^{-\frac{y^2}{2r_0^2}} e^{-\frac{z^2}{2r_0^2}} \quad (141)$$