Leo Radzihovsky, December 12, 2005

PHYS 5250: Quantum Mechanics - I

FINAL EXAM

Note: Please be as explicit as possible. Generous partial credit will be given for correct approach, even for wrong final answers, if you can convince me that you know what you are doing.

Unless explicitly requested, you are not required to derive things from scratch (particularly when I say, "What is...?", as opposed to "Derive..."). However, if you are indeed not deriving your answer from scratch (e.g., because you simply know it), please, at least say some brief words about how you are (maybe mentally) obtaining the answer. For example "This Schrodinger's equation can be solved by separation of variables introducing ..., which then reduces it to a problem that is identical to the one that we studied in class and gives Laguerre polynomials as eigenstates.". Of course at the same time please keep in mind, that the fewer in-between steps and explanations you give, the more difficult it will be for me to give you partial credit for conceptual understanding, *if* you make a mistake.

Please check your answers carefully!

Total Points: 100.

Good Luck!

1. (30 points) Harmonic Oscillator

   Consider a particle free to move in three dimensions, of mass m and charge q and subjected to a 1D harmonic potential $U = \frac{1}{2}m\omega^2 x^2$, and sitting in its ground state. At $t = 0^+$, an uniform, constant electric field $\mathbf{E} = E\hat{\mathbf{x}}$ is suddently turned on. Answer the following clearly and explicitly, indicating all the relevant quantum numbers and coordinate dependences:

   (a) (4 pts) Write down the ground-state wavefunction for this 3D problem right before the E-field is turned on.

   (b) (4 pts) Write down the Hamiltonian for the particle after the electric field is turned on, and give corresponding eigenstates and the energy spectrum for this 3D problem.

   (c) (5 pts) Compute the probability P_0 of finding the system in its ground state at time $t = 0^+$.

   (d) (7 pts) Compute the probability P_e of finding the system in one of its (particular) excited states at time $t = 0^+$, clearly indicating the dependence on all the quantum numbers characterizing the excited states. If you are having difficulties to compute the final answer explicitly, at least write down the formal expression in terms of the answers to questions above.

   (e) (10 pts) Find the wavefunction $\psi(\mathbf{r}, t)$, for $t > 0$.

   You might find the following useful:
   $\int_{-\infty}^{+\infty} dx H_n(x) H_m(x) e^{-x^2} = \pi^{1/2} 2^n n! \delta_{n,m}$, where the Hermite polynomials have a convenient generating function $e^{-s^2+2xs} = \sum_n \frac{s^n}{n!} H_n(x)$.

   Hint: Note that this last generating function is actually equivalent to a decomposition of a coherent state in terms of the oscillator eigenstates. In order to take advantage of this (or related) relation you will find it useful to make a couple of simple variable manipulations.

2. (25 points) One-dimensional Molecule

   Consider an electron in a double square-well potential, modelling the attractive potential of two ions of a diatomic molecule (e.g., singly ionized H_2^+). Let's for simplicity model the wells by δ-functions, so the total attractive potential is $U(x) = -U_0(\delta(x + a) + \delta(x - a))$.

   (a) (10 pts) Find the molecule's ground state, determining the spatial wavefunction completely, except for the overall normalization (that you do not need to find) and one other parameter κ, that is related to the ground state energy E_0.

   (b) (10 pts) Write down (but of course do not solve) the transcendental equation satisfied by this one parameter κ, and use its general form (e.g., graphically) to show that a bound state solution κ_* always exists, independent of how weak the strength of the attractive potential U_0 is.

(c) (5 pts) What is the ground-state energy E_0 in terms of the solution of the above transcendental equation?

Hint: It is convenient to define your zero of energy by where the potential is absent. Also, you might find it useful to note that the parity operator commutes with this Hamiltonian.

3. (15 points) Evolution operator

Compute the time-evolution operator in coordinate representation $U(x_f, x_0; t)$ for a free particle of mass m

(a) (8 pts) directly from its definition in terms of the Hamiltonian, by computing the corresponding coordinate matrix element, and

(b) (7 pts) by using path integral method. In this second case, you do not need to compute the (x_f, x_0-independent) prefactor, but should say a few words on how you would go about doing it.

4. (20 points) Two identical spin 1/2, mass m fermionic particles are interacting via an attractive $U(|\mathbf{r}_1 - \mathbf{r}_2|) = -\alpha/|\mathbf{r}_1 - \mathbf{r}_2|$ two-body potential. Using a clearly stated list of logical steps relate this two-body problem to one that we have studied in great detail. Thereby write down explicitly the ground state energy and the full wavefunction for this two spinful fermion system, expressing them in terms of m and α, and spin projection quantum numbers and coordinates $\mathbf{r_1}$ and $\mathbf{r_2}$.

Note: please ignore any direct spin-spin and/or spin-orbit interactions.

5. (10 points) Find the spectrum of a rotor-model Hamiltonian $H = \frac{J_x^2}{2I_x} + \frac{J_y^2}{2I_y} + \frac{J_z^2}{2I_z}$ in the $j = 1/2$ representation of the angular momentum, i.e., for the eigenvalue of J^2 being $\hbar^2 j(j+1)$, with $j = 1/2$. Above I_i's are moments of inertia of the rotor, that you can take to be simply constants.