Leo Radzihovsky, December 19, 2006

PHYS 5250: Quantum Mechanics - I

FINAL EXAM

Note: Please be as explicit as possible. Generous partial credit will be given for correct approach, even for wrong final answers, if you can convince me that you know what you are doing.

Unless explicitly requested, you are not required to derive things from scratch (particularly when I say, "What is...?", as opposed to "Derive..."). However, if you are indeed not deriving your answer from scratch (e.g., because you simply know it), please, at least say some brief words about how you are (maybe mentally) obtaining the answer. For example "This Schrodinger's equation can be solved by separation of variables introducing ..., which then reduces it to a problem that is identical to the one that we studied in class and gives Laguerre polynomials as eigenstates.". Of course at the same time please keep in mind, that the fewer in-between steps and explanations you give, the more difficult it will be for me to give you partial credit for conceptual understanding, *if* you make a mistake.

Please check your answers carefully!

Total Points: 100.

Good Luck!

1. (25 points) Quickies

   (a) (10 points) Show that if two observables $\hat{A}$ and $\hat{B}$ do not commute with each other but each commutes with a Hamiltonian $\hat{H}$, i.e., $[\hat{A}, \hat{B}] \neq 0$, $[\hat{H}, \hat{A}] = 0$, $[\hat{H}, \hat{B}] = 0$, then this generally implies that the spectrum of the Hamiltonian is degenerate.

   (b) (7 points) In quantum mechanics, in terms of a unitary transformation operator $\hat{U}_\alpha$ what is a statement of the corresponding symmetry of a system governed by a Hamiltonian $\hat{H}$, and what is its main consequence? After a general statement, illustrate with an example of e.g., translational symmetry.

   (c) (8 points) Show that a state $|\alpha\rangle = e^{i\hat{x}\alpha/\hbar}|p\rangle$ (where $|p\rangle$ is an eigenstate of momentum operator $\hat{p}$ with eigenvalue p) is also an eigenstate of $\hat{p}$ and derive its eigenvalue α in terms of p.

2. (50 points) A particle in a linear potential (e.g., electron in an electric or a gravitational field)

   (a) (5 points) Is the spectrum of a particle of mass m in a linear potential $V(x) = fx$ discrete or continuous?

   (b) (10 points) Find the eigenstates $|\psi_E\rangle$ for a given energy E in momentum representation, not worrying too much about their normalization.

   Hint: Since $\hat{p}$ and $\hat{x}$ appear on equal footing in the canonical commutation relation, for this problem it is convenient to use a momentum representation directly.

   (c) (10 points) Write down a formal expression for the corresponding coordinate representation of $|\psi_E\rangle$, i.e., for the wavefunction $\psi_E(x)$ and show that it is a function of a dimensionless variable $\zeta = (x - x_E)/a_f$. What are lengths x_E and a_f and what is their significance?

   Hint: This wavefunction is proportional to the so-called Airy function, $Ai(\zeta)$.

   (d) (15 points) If in addition a particle experiences an impenetrable potential (hard wall) at $x = 0$, confining it to $x > 0$, what is the resulting spectrum? Deduce the characteristic length scale x_n (expressing it in terms of a characteristic length found above) and the qualitative nature (upto dimensionless factors) of the corresponding spectrum via very simple and rough semi-classical (qualitative) arguments (of the type used in class and on the homework) of balancing kinetic and potential energies, i.e., by minimizing the total energy subject to the uncertainty relation constraint.

   (e) (10 points) Write down the spectrum in terms of zeros α_n of the Airy function defined by $Ai(\alpha_n) = 0$. Sketch a few lowest eigen-wavefunctions $\psi_E(x)$.

3. (25 points) Consider a diatomic molecule consisting of two identical, neutral, spin-1/2, fully spin-polarized fermionic atoms bound by a potential $U(|\mathbf{r}_1 - \mathbf{r}_2|)$. Treating atoms as point particles:

(a) (12 points) Write down a general two-particle wavefunction $\Psi_{\alpha_1,\alpha_2,\ldots}(\mathbf{r}_1,\mathbf{r}_2)$ appropriate for this system, identifying allowed quantum numbers $\vec{\alpha}$ that uniquely characterize such a state, and clearly but briefly justifying your answer.

(b) (13 points) Within a harmonic binding potential U approximation, write down the explicit two-atom ground-state wavefunction, identifying the degeneracy of this ground-state subspace and justifying your answer.

Hint: $Y_{\ell,m}(-\hat{\mathbf{r}}) = (-1)^\ell Y_{\ell,m}(\hat{\mathbf{r}})$; do not worry about proper normalization.