Leo Radzihovsky, December 17, 2007

PHYS 5250: Quantum Mechanics - I

FINAL EXAM

Note: Please be as explicit as possible; points will be deducted for answers that are correct but do not show any work. Generous partial credit will be given for a correct approach, even if the final answer is not quite correct, if you can convince me that you know what you are doing.

Unless explicitly requested, you are not required to derive things from scratch (particularly when I say, "What is...?", as opposed to "Derive..."). However, if you are indeed not deriving your answer from scratch (e.g., because you simply know it), please, at least say some brief words about how you are (maybe mentally) obtaining the answer. For example "This Schrodinger's equation can be solved by separation of variables introducing ..., which then reduces it to a problem that is identical to the one that we studied in class and gives Laguerre polynomials as eigenstates.". Of course at the same time please keep in mind, that the fewer in-between steps and explanations you give, the more difficult it will be for me to give you partial credit for conceptual understanding, *if* you make a mistake.

Please check your answers carefully!

Total Points: 100.

Good Luck!

1. (40 points) Quickies (with brief explanations)

   (a) (9 points) A system's J^2 and J_x operators are measured and found to have eigenvalues $3\hbar^2/4$ and $+\hbar/2$, respectively, (i) if J_z is subsequently measured once, what value(s) can be found? (ii) what is the probability of finding $J_z = +\hbar/2$, right *after* the above J_x measurement? (iii) what is the expectation value of J_z, right *after* the above J_x measurement?

   (b) (5 points) What is the unitary operator U_{ϕ_0} acting in Hilbert space that executes rotation in 2D by angle ϕ_0? Write its coordinate representation and demonstrate explicitly that when acting on a wavefunction $\psi(\phi)$ it produces a state of a system rotated by ϕ_0.

   (c) (8 points) (i) Write down a (formal) path-integral representation for a wavefunction $\psi(x,t)$ at time t for a Hamiltonian $H = p^2/2m + V(x)$ and $t = 0$ wavefunction $\psi_0(x)$, clearly indicating all the limits. (ii) Give the explicit coordinate-based answer in terms of $\psi_0(x)$ (based on whatever you like, memory, hints from the physically correct form and units, quick derivation using (i), etc...) for the case $V(x) = 0$.

   Comment: I am looking for an explicit and detailed expression (with everything clearly labelled and defined), but of course without any attempt to evaluate the path integral and the formal expression in (i) for $V(x) \neq 0$.

   (d) (10 points) A harmonic oscillator (defined by frequency ω_0) at time $t = 0$ is in a coherent state $|z\rangle$. (i) What state does it evolve into at a subsequent time t? (ii) What is the expectation value of $\hat{a}^2$ at time t in this state ($\hat{a}$ is annihilation operator.)? (iii) Answer (ii) via Heisenberg representation instead.

   Comment: For convenience, you can ignore the vacuum energy, i.e., define the zero of the energy to be the ground state energy.

   (e) (8 points) A spin 1/2 system is described by a density matrix

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}. \tag{1}$$

   (i) Does this density matrix represent a pure or a mixed state? How do you know?

   (ii) Use ρ to compute the expectation value of $\vec{S}$.

2. (30 points) Bose-Einstein condensate trapped in a harmonic potential

   (a) (5 points) Write down the corresponding (i) Hamiltonian, H (ii) ground state N-particle wavefunction, $\Psi_{GS}(x_1, x_2, ..., x_N)$ and (iii) the total energy E_{GS} of N *non*interacting bosonic atoms trapped in a harmonic potential characterized by frequency ω_0?

   Comment: Although you need not do any technical calculations for this part, please do provide *brief* comments as you take steps to deduce your answer.

(b) (8 points) If the trap frequency is *suddenly* reduced down to ω_1, compute the probability of finding the N particle system in its (i) ground state for the new trap? (ii) first excited state for the new trap, right after the shift?

Comment/Hint: Clearly, it is important to use a properly *normalized* wavefunction. As one would expect, in (i) the probability should go to zero as $\omega_1/\omega_0 \to 0$.

(c) (7 points) Now, for a single particle (of mass m) starting in the ground state of such trap, what is the probability density $P_v(t)$ at time t of finding the particle with velocity v after (at time $t = 0$) the trap is suddenly and completely shut off?

(d) (10 points) For part (c), compute the wavefunction $\psi(x, t)$ and the corresponding probability density $P(x, t)$ after a subsequent evolution at time $t > 0$?

3. (30 points) Quantum Hall effect in a 1D parabolic quantum well

(a) (25 points) Derive the spectrum and eigenstates of a charged particle confined to move in 2D xy-plane in a presence of a constant magnetic field $\vec{B} = B\hat{z}$ and a one dimensional harmonic trap $V(x) = \frac{1}{2}m\omega_0^2 x^2$.

Comment: It is extremely helpful to pick a convenient gauge in describing the effects of the magnetic field.

(b) (5 points) Make a physical plausability argument for the nature of the spectrum, as for example compared to the problem without the additional harmonic potential studied in class, and sketch the spectrum.