Leo Radzihovsky, Fall 2007

PHYS 5250: Quantum Mechanics - I

Homework Set 1

Issued August 27, 2007
Due September 10, 2007

Reading Assignment: Sakurai, Ch.1.1, Shankar, Ch. 2, 3; Schiff Ch. 1

1. Use (i) Lagrangian, (ii) Hamiltonian formalism to derive Newton's equations for a two-dimensional (2d) isotropic harmonic oscillator in

   (a) cartesian coordinates

   (b) polar coordinates

   showing that they are consistent with each other.

   Using Poisson brackets, verify that the canonical momenta L_ϕ associated with the angle ϕ is a constants of motion, i.e., is conserved.

2. Using the Lagrangian for a 1d harmonic oscillator, its action functional $S[x(t), \dot{x}(t)]$, and the explicit general solution $x_{cl}(t)$ of the corresponding equation of motion, derive the action $S(x_f, x_i, T)$, as a function of the initial ($x_i \equiv x(t_i)$) and final ($x_f \equiv x(t_f)$) locations of the particle and time duration $T = t_f - t_i$.

3. Consider the action for a noninteracting nonrelativistic particle of mass m given by:

   $$S[\overline{\psi}(\mathbf{r},t), \psi(\mathbf{r},t)] = \int d^3r dt \left[-i\hbar\overline{\psi}\partial_t\psi + \frac{\hbar^2}{2m}\nabla\overline{\psi}\cdot\nabla\psi + V(\mathbf{r})\overline{\psi}\psi \right] \quad (1)$$

   Treating $\overline{\psi}(\mathbf{r},t)$ and $\psi(\mathbf{r},t)$ as independent fields, derive the corresponding Euler-Lagrange equations, i.e., equations corresponding to $\frac{\delta S}{\delta\overline{\psi}(\mathbf{r},t)} = 0 = \frac{\delta S}{\delta\psi(\mathbf{r},t)}$.

   Hint: (a) Functionals and calculus of variation straightforwardly extends to functions of many variables, such $\psi(\mathbf{r},t)$, by thinking of each variable $(\mathbf{r},t)$ as a discrete label. (b) Ignore the boundary terms.

4. Using Bohr-Sommerfeld quantization, whose simplified version is that angular momentum in a circular orbit is quantized according to $L = mvr = n\hbar$, estimate the spectrum E_n and radius r_n of a hydrogenic atom with atomic number Z.

   Make the units of E_n and r_n explicit by expressing your answer in terms of the electron's Compton wavelength $\lambda_e \equiv h/mc$ and the fine structure constant $\alpha \equiv e^2/\hbar c$.

   Hint: ignore electron-electron interaction and only consider circular stationary "orbits" of a single electron.

5. Using a combination of Heisenberg uncertainty principle and Bohr-Sommerfeld quantization for the nth eigenstate ($p_n r_n \approx n\hbar$) estimate the spectrum E_n and the extent r_n of the corresponding eigenstates for the following class of bound state problems:

   (a) a harmonic oscillator with $V(x) = \frac{1}{2}m\omega^2 x^2$,

   (b) a "quartic" oscillator with $V(x) = \frac{1}{4}ax^4$,

   (c) a hydrogenic atom with Coulomb potential $V(r) = -Ze^2/r$,

   (d) an electron in a potential $V(x) = \frac{1}{s}V_0(x/x_1)^s$. In this latter case also study the spectrum for large $s \gg 1$ and explain the result to which E_n reduces.

   Hint: Minimize the corresponding total energy subject to above quantum mechanical constraint.

6. Show (as expected, given that it represents conserved matter) that the probability distribution $P(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2$ satisfies a continuity equation $\partial_t P + \nabla \cdot \mathbf{J} = 0$.

   What is the corresponding particle current $\mathbf{J}$?

   Hint: Take advantage of the fact that $\psi(\mathbf{r}, t)$ satisfies the Schrodinger's equation.