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PHYS 5250: Quantum Mechanics - I

Homework Set 5

Issued October 22, 2007
Due November 5, 2007

Reading Assignment: Shankar: Chs. 10, 11; Sakurai: Secs. 1.6, 1.7, Chs. 4, 6

1. Using path integral formulation, derive a *free* particle time evolution operator in coordinate representation, $U_0(x, x'; t, 0)$. Rather than following shortcuts we discussed in class, please do this by explicitly evaluating $N-1$ gaussian integrals that arise in discretizing the path integral for $U(x, x'; t, 0)$.

   Hint:

   The most direct route is to first demonstrate and then utilize the "closure" property of a propagator, namely to show that $U_0(x, x''; t, t'') = \int_{-\infty}^{\infty} dx' U_0(x, x'; t, t') U_0(x', x''; t', t'')$. To show this you will simply need to do a single gaussian integral, taking advantage of the calculus that we developed in class. There are other, less direct but technically simpler ways of doing it. Can you find one or two other ways of doing it?

2. Consider a Hamiltonian of two interacting particles $H = p_1^2/2m_1 + p_2^2/2m_2 + V(\mathbf{r}_1 - \mathbf{r}_2)$ moving in 3d.

   (a) Show that by performing a unitary transformation to the center of mass $\mathbf{R}_{cm}, \mathbf{P}_{cm}$ and relative $\mathbf{r}, \mathbf{p}$ coordinates, above Hamiltonian decouples into $H = H_{cm} + H_{rel}$, where $H_{cm} = P_{cm}^2/2M$ and $H_{rel} = p^2/2\mu + V(\mathbf{r})$ are the center of mass and relative coordinate Hamiltonians, where M and μ are the total and reduced masses respectively whose expressions in terms of m_1 and m_2 you should derive.

   (b) Verify that this transformation is indeed unitary, i.e., that the commutation relations between $\mathbf{R}_{cm}$ and $\mathbf{P}_{cm}$, and between $\mathbf{r}$ and $\mathbf{p}$ remains canonical. Do this by finding the relation between old and new variables and then computing the commutation relations between all *new* variables ($[R_{cm}^i, p^j]$, $[R_{cm}^i, P_{cm}^j]$, etc.) using commutation relations for the old variables.

   (c) Take advantage of the above separability of H to find the $\mathbf{R}_{cm}$ dependence of the eigenstates of H, and derive the effective Schrodinger equation satisfied by the part of the wavefunction that only depends on the relative coordinate $\mathbf{r}$.

(d) Considering a physical case in which $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|)$ (i.e., that the interaction is rotationally invariant, depending only on the distance between the two particles), show that the Schrodinger equation for $\psi_{rel}(\mathbf{r})$ can be further separated in a spherical coordinate system by reducing to three independent eigenvalue differential equations for the radial, polar and azimuthal part respectively dependent on r, θ, and ϕ.

3. When an energy measurement is made on a system of three spinless bosons in a box, the n values obtained were 3, 3, and 4. Write down a symmetrized, normalized state vector.

4. Imagine a situation in which there are three particles and only three states a, b, and c available to them. What is the total number of allowed, distinct configurations for this system if the particles are:

   (a) labeled, i.e., distinguishable

   (b) indistinguishable bosons

   (c) indistinguishable fermions

5. Bloch theorem and discrete translational invariance

   Consider a particle in a periodic potential with period a. With the particle an electron and the periodic potential due to a positive ions arranged in a periodic arrangement in a crystalline solid, this problem is at the very heart of much of modern solid-state physics.

   (a) Show explicitly that the corresponding Hamiltonian commutes with a translation operator T_ϵ only for translations that are a multiple of a, i.e., $\epsilon = na$, $n \in \mathcal{Z}$.

   (b) Use above result together with the observation that T_a is a unitary operator to clearly argue that the eigenstates of H are (the so-called) Bloch states, given by $\psi_{k,n}(x) = e^{ikx}u_{k,n}(x)$, where k is a continuous (for an infinite system) quantum number limited to a finite range (called 1st Brillouin zone) that can be taken to be $0 < k < 2\pi/a$ and $u_{k,n}(x)$ is periodic part of the wavefunction with period a, labeled by k and another discrete quantum number n ($\in \mathcal{Z}$).

   (c) Derive an effective Schrodinger equation satisfied by $u_{k,n}(x)$.

   (d) What can you say about the property of the energy eigenvalues $E_n(k)$?

   Note that althought the eigenfunctions are *not* periodic with period a, as expected on physical grounds the probability density $P_{k,n}(x) = |\psi_{k,n}(x)|^2$ of finding a particle at positive x is. This is no different than in a special case of this problem of a constant $V(x)$ (e.g., 0), where the eigenfunctions are plane waves e^{ikx} that are *less* symmetric (i.e., not translationally invariant) than the Hamiltonian.

6. Compute the ground state energy of system of N identical particles confined to a single infinite square-well potential of width L, when the particles are

   (a) Bosons

   (b) Fermions

   Suggestion: to obtain the final answer you should find it useful to replace a sum by an integral valid for large N and L. Note that the answer for bosons is extensive while for fermions is “super-extensive”, i.e., scaling with the number of particles much faster than the total number of particles in the system.

7. Please think about (but do *not* have to do) Shankar’s problems 10.3.4, 10.3.5 and 10.3.6.