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PHYS 5250: Quantum Mechanics - I

**Homework Set 6**

Issued November 5, 2007
Due November 26, 2007

Reading Assignment: Shankar, Ch. 12; Sakurai, Ch.3.1-3.6

1. Show that any function $H(x)$ of a scalar product $x = \mathbf{E}_1 \cdot \mathbf{E}_2$ of two arbitrary vector operators $\mathbf{E}_1$, $\mathbf{E}_2$ commutes with all components of the angular momentum operator $\mathbf{L}$. This thereby demonstrates explicitly that a rotationally-invariant Hamiltonian (i.e., one that consists only of scalar products of vector operators) commutes with the angular momentum vector.

2. Show that the z-component of the angular momentum operator $L_z = -i\hbar(x\partial_y - y\partial_x)$ can be written as $L_z = -i\hbar\partial_\phi$, where $\phi$ is the azimuthal angle of the spherical coordinate system.

3. Three identical, elementary bosonic particles of mass $m$ are confined to move on a circle of radius $r$ and are rigidly fixed to form an equilateral triangle, so that their only degree of freedom is the common azimuthal angle $\phi$. Ignoring spin, compute the wavefunction and the corresponding energy spectrum.

   Hint: Think about the symmetry of the eigenfunctions $\psi_E(\phi)$.

4. Express lowest six spherical harmonics $Y_{\ell,m}(\theta, \phi)$ in Cartesian coordinates, $x, y, z$.

5. (a) A particle is described by a wavefunction $\psi(\rho, \phi) = Ae^{-\rho^2/\rho_0^2}\cos^2\phi$. By expressing $\psi$ in terms of eigenstates of $L_z$, show that probability $P(\ell_z)$ of finding the particle with the $z$ component of angular momentum equal $\ell_z$ is $P(0) = 2/3$ and $P(\pm 2\hbar) = 1/6$ and zero for all other values of $\ell_z$.

   (b) Another particle is described by a wavefunction $\psi(x, y, z) = A(xy+yz+xz)e^{-r/r_0}$. What is the probability of finding the particle with a particular value of its square of angular momentum? If the particle is found to have $\ell = 2$, what are the relative probabilities of finding this particle with $m = \pm 2, \pm 1, 0$?

6. (Shankar 12.5.14) Rotation of $\ell = 1$ eigenstates

7. Harmonic Oscillator

(a) Study a two-dimensional harmonic oscillator in polar $(\rho, \phi)$ coordinates

i. Find the energy spectrum and the corresponding eigenstates. For the eigenstates it is sufficient to establish the series solution to the corresponding differential equation and to determine the recursion relation for the series-solution coefficients $c_i$.
Hint: To help guide you, for the steps outlined in problem 12.3.7 of Shankar.

ii. Show the spectrum $E_n$ by marking (with a short horizontal dashes) energy levels for the lowest 10 states labelled by $n, m$, where $n$ is the principle (radial) quantum number and $m$ the eigenvalue of the z-component of the angular momentum $L_z/\hbar$. Make the horizontal axis the m-axis, so that two states with the same value of $n$ but different value of $m$ are shifted horizontally relative to each other, with each appearing at its correct value of $m$ along the horizontal axis.

iii. Deduce the degeneracy of energy $E_n$, confirming that it agrees with that obtained solving this problem via Cartesian coordinates $x, y$.

iv. Write down explicit wavefunctions $\psi_{n,m}(\rho, \phi)$ for the lowest eigenstates $(n = 0, m = 0)$, $(n = 1, m = \pm 1)$, $(n = 2, m = 0, \pm 2)$, and verify explicitly that each of these are linear combinations of corresponding degenerate eigenstates $\psi_{n_x,n_y}(x, y) = H_{n_x}(x)e^{-x^2/2}H_{n_y}(y)e^{-y^2/2}$. Please make this connection explicit.
Hint: In establishing the relation, please do not worry about any normalization constants.

(b) Above Hamiltonian is a good model for a noninteracting (atoms do not see each other, only the harmonic trap potential) atomic gas confined to a harmonic trap potential. Consider what happens if one rotates such a gas at frequency $\Omega$, as is now routinely done by our colleagues (maybe even some of your classmates) in JILA. The corresponding Hamiltonian for the rotating system is given by an addition of a single term, $-\Omega L_z$ to the Hamiltonian above. Find the energy spectrum for such rotating atomic gas and plot it the way you did above, doing it for $\Omega < \omega$. What do you think happens when $\Omega \to \omega$ and faster rotation?

Hint: Having found the spectrum for the stationary gas, you should simply be able to write down the spectrum for the rotating one without doing any calculations. Regarding $\Omega \to \omega$ limit, think about what happens for a corresponding classical problem of a particle trapped in a quadratic (harmonic) potential with frequency $\omega$, and rotated with angular frequency $\Omega$.

(c) Guided by the discussion of the last section of Ch. 12 in Shankar (also see problem 12.6.11) for the 3d harmonic oscillator solved in spherical coordinates and characterized by quantum numbers $n, \ell, m$, answer the following:

i. Compute the degeneracy of the $n$-th energy level of a 3d harmonic oscillator, confirming that it agrees with the result you found on a previous homework using Cartesian coordinates $D_n^{3d} = (n + 1)(n + 2)/2$.

ii. Using results of discussion in Shankar (but without performing any explicit series solution calculations), write down four lowest eigenfunctions $\psi_{n,\ell,m}(r,\theta,\phi)$ in spherical coordinates, and then demonstrate (in a way very similar to that for 2d case, above) an explicit connection of these states to wavefunctions in Cartesian coordinates, $\psi_{n_x,n_y,n_z}(x,y,z) = H_{n_x}(x)e^{-x^2/2}H_{n_y}(y)e^{-y^2/2}H_{n_z}(z)e^{-z^2/2}$. Hint: For $\ell = n$ all expansion coefficients $c_i$ (in the series solution) vanish except the lowest one, $c_0$. In establishing the relation, please do not worry about any normalization constants.